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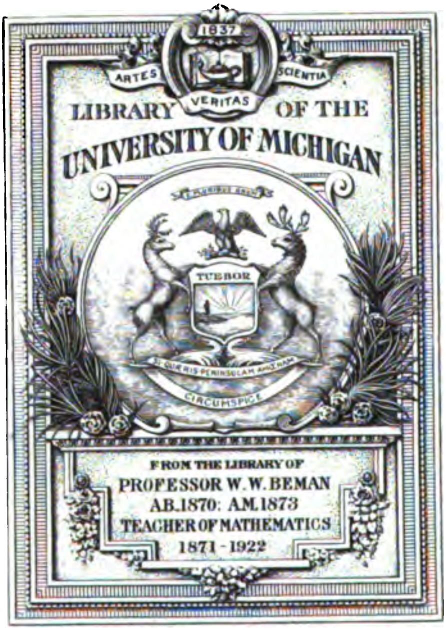

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GEOMETRICAL ANALYSIS,

AND

GEOMETRY OF CURVE LINES,

BEING

VOLUME SECOND

OF

A COURSE OF MATHEMATICS,

AND DESIGNED AS

AN INTRODUCTION TO THE STUDY OF
NATURAL PHILOSOPHY.

BY
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PREFACE.

The present work, which forms the Second Volume of a Course of Mathematics, is the fruit of persevering application. Owing to various accidents, it has been repeatedly interrupted and resumed; and I feel now relieved by the discharge of a task, which nothing but the anxious desire to promote a juster taste in the cultivation of Mathematical Science could have induced me to undertake.

This volume includes three distinct treatises, which may be thus enumerated.

I. Geometrical Analysis.—This tract, in a less finished state, was annexed to the first and second editions of the Elements of Geometry. It consisted of a series of choice Problems, rising in gradation, and spreading into the rich and ample fields of the Ancient Analysis. In collecting, disposing, and sometimes framing the materials, I spared no exertion. The Essay was accordingly well received both at home and abroad, and has conspired to advance the study of Geometry, by reviving the fine models bequeathed by the Greeks. It has been translated into the French and German languages, an honour which very few of our mathematical works have obtained.

In finally committing this Treatise to the Public, I have endeavoured to render it as complete as possible. I have carefully revised the whole, and pruned some excrescences; but I have filled up other important parts, and extended considerably the chain of propositions. The study of such a digest appears admirably fitted to improve the intellect, by training it to habits of precision, arrangement, and close investigation. The spirit of Geometrical Analysis may be carried with the happiest effect into the domains of Inductive Philosophy, which are explored by a similar procedure.

II. GEOMETRY OF LINES OF THE SECOND ORDER.—These curves, discovered by the immediate successors of Plato, drew their origin from the section of a plane perpendicular to another which touched the side of a regular cone, their different species being determined by the angle of its apex. The Parabola was formed by the section of a right-angled cone, and the Ellipse and Hyperbola generated by a like section of the acute and of the obtuse cones. Apollonius showed that the same curves would be produced from a regular cone of any angle, if dissected by planes at different inclinations. The moderns have extended the property to the oblique cone, or the cone defined by radiants from a fixed point or apex to a circular base, which, being cut in various ways by the same plane, gives the several species of the curve.

But although the dissection of the cone gave birth to these lines, they still decidedly belong to Plane Geometry. Several eminent authors, however, have preferred the mode of deducing their properties from the solid. Yet, notwithstanding some incidental advantages, the difficulty of representing or conceiving the intersections of planes, perplexed too by a multiplicity of lines, renders the progress of the student extremely tedious and irksome. The author himself appears always glad in escaping as soon as possible from that intricate path, to follow the smooth road of Plane Geometry.

The direct and luminous method of treating these curves, therefore, is, to transfer them at once to a plane surface, selecting as a definition some simple property from which the other properties are the most easily evolved. The adoption of a generic character, instead of a definition of each species of curve, likewise abridges greatly the train of demonstration, and, while it preserves the unity of design, it affords all the lights of contrast.

I have chosen as the Genesis or Fundamental relation of the Lines of the Second Order, the beautiful property noticed by Pappus in his *Mathematical Collections*, and investigated by Newton himself in his *Arithmetica Universalis*,—that the distances of any point in the curve from the focus and from the directrix, have always a constant ratio. The celebrated Boscovich was the first who composed a Treatise of Conic Sections on this principle in 1744: It consisted of only a few propositions, but drawn out into a string of corollaries. After the lapse of half a century, Mr Newton of Cambridge condensed and modified that excursive composition into a system, which has the merit of being clear, neat, and concise. Nearly about the same time, Mr Walker of Nottingham, apparently without any knowledge of what Boscovich had done, fell upon a similar mode of investigation; but his Conic Sections, of which the first part was printed in 1795 in a quarto volume, though ingenious and strictly geometrical, is unfortunately so prolix and ponderous, as to damp the ardour of the most resolute student.

In the composition of the treatise now laid before the Public, I have sought to avoid the opposite extremes of brevity and diffusion; and pursuing an uniform plan, I have endeavoured to exhibit, in a full light, the properties and mutual relations of those curves. The multifarious labours of former writers have left scarcely any room for invention; but the learned reader will perceive the pains I have taken in arranging the materials, and in improving the simplicity, clearness, and elegance of the demonstrations. The common principle which unites those curves, has enabled me to reduce into a very moderate compass their leading properties. The First Book comprises a more copious assemblage of interesting propositions than are usually found in elementary systems. But the Second Book contains a selection of the higher properties, which peculiarly claim the attention of those who cultivate dynamics and physical astronomy.

III. GEOMETRY OF THE HIGHER CURVES.—A Treatise formed on a regular plan, to embrace the chief properties of all the remarkable curves above the Lines of the Second Order, has long been wanted, for completing the course of mathematical instruction. Some works, indeed, on Conic Sections, have bestowed a glance over this subject; but their notices are scanty and confined to a very few curves. The properties of the higher curves lie scattered through volumes of difficult access, and are only brought occasionally into view as exemplifications of the rules of the method of fluxions or of the differential and integral calculus. But the beautiful relations of those curves expand our prospects, and afford wide scope for the application of a refined geometry. To avoid circuitous demonstration, it became expedient, on this occasion, to depart somewhat from the ancient manner of proceeding; but such deviations nowise impair the accuracy of the reasoning. I have only followed up the successive improvements of Roberval, Cavalleri, Des Cartes, and Barrow. The principles applied, are in fact substantially the same as those of the differential and integral calculus, which really derives its main advantage from its algorithm, or that clear and compact form of notation invented by Leibnitz, and improved on the Continent by his followers, the Bernoullis, Euler, and Lagrange.

The superior elegance and perspicuity with which the geometrical process unfolds the properties of those higher curves, may show, that the fluxionary calculus should be more sparingly employed, if not reserved for the solution of problems of a more arduous nature. I have drawn the materials from various sources, but chiefly from the writings of Huygens and the two Bernoullis. But the value of the treatise will consist in the symmetry of the structure, and the beauty and importance of the propositions which it has combined. In filling up the outline, it may be perceived that I have been led to make some contributions to the common stock.

It only remains, for completing my plan, to produce a volume on Descriptive Geometry and the Theory of Solids, comprehending Perspective, the Projection of the Sphere, and Spherical Trigonometry. But, in the meantime, I must devote my whole attention to the composition of a work of greater urgency and of higher interest—The Elements of Natural Philosophy.

COLLEGE OF EDINBURGH,
Oct. 30. 1821.

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GEOMETRICAL ANALYSIS.

BOOK I.

DEFINITIONS.

1. *Quantities* are said to be *given*, which are either exhibited, or may be found.

2. A *ratio* is said to be *given*, when it is the same as that of two given quantities.

3. *Points*, *lines*, and *spaces*, are said to be *given in position*, if they have always the same situation, and are either actually exhibited, or may be found.

4. A *circle* is *given in position*, when its centre is given: it is *given in magnitude*, if its radius be given.

5. *Rectilineal figures* are said to be *given in species*, when figures similar to them are given.

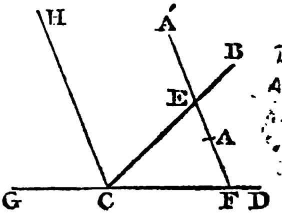
PROP. I. PROB.

Through a given point, to draw a straight line at equal angles with two inclined straight lines given in position.

Let A be the given point, and CB, CD the inclined straight lines which are given in position.

ANALYSIS.

Let those inclined lines meet in C; draw (I. 23. El.) CH parallel to FE, and produce DC. The exterior angle GCH (I. 22. El.) is equal to CFE, and ECH is equal to the alternate angle CEF; but the angle CFE is equal to CEF, and consequently GCH is equal to ECH, and the angle GCE is thus bisected by the straight line CH. Wherefore (I. 5. El.) CH is given in position, and hence (I. 23. El.) the parallel EF is also given.



COMPOSITION.

Bisect (I. 5. El.) the adjacent angle GCB by the straight line CH, and parallel to this draw EF (I. 23. El.) through the given point A; the angle CEF is equal to CFE. For these angles are equal respectively to the exterior and to alternate angles GCH and ECH of the parallels CH and AF, and are consequently equal to each other.

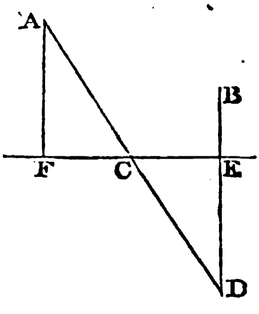
It is obvious, that if BC and CD become less inclined, GCH, half the exterior angle, will approach to a right angle; whence, in the extreme limit of parallelism, HC, and therefore AF, will be perpendicular to CD or CB.

PROP. II. PROB.

Through a given point, to draw a straight line, such that the segments intercepted by perpendiculars let fall upon it from two given points, shall be equal.

The points A, B, and C being given,—to draw a straight line FE through C, so that the parts CF and CE, cut off by the perpendiculars AF and BE, let fall from A and B, shall be equal.

ANALYSIS.



Produce AC to meet the extension of BE in D. The right angled triangles AFC and DEC, having the vertical angle ACF equal to DCE, and the side CF by hypothesis equal to CE, are (I. 20. El.) equal, and hence the side CA is equal to CD. But CA is evidently given; wherefore CD and the point D are given; BD is consequently given, and hence the line CE at right angles to it is given.

COMPOSITION.

Join AC, and produce it till CD be equal to it; join BD, on which, from C, (I. 6. El.) let fall the perpendicular FCE, which is the line required. For, draw AF parallel to BD, and the triangles FAC and EDC, having the angles ACF, AFC equal to DCE, DEC, and the side AC equal to CD, (I. 20. El.) are equal, and consequently CF is equal to CE.

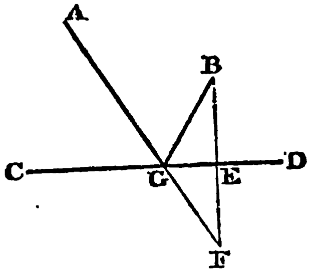
PROP. III. PROB.

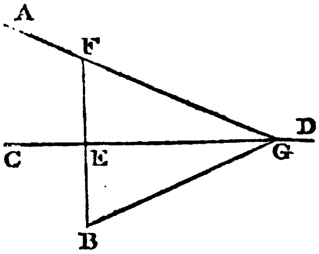
From two given points, to draw straight lines, making equal angles at the same point in a straight line given in position.

Let A, B be two given points, and CD a straight line given in position; it is required to draw AG and GB to some point in it, so that the angles AGC and BGD on either side shall be equal.

ANALYSIS.

From B, one of the given points, let fall the perpendicular BE, and produce it to meet AG, or its extension in F. The angle BGE, being equal to AGC, is equal to the vertical angle FGE, the right angle BEG is equal to FEG, and the side GE is common to the triangles GBE and GFE, which (I. 20. El.) are therefore equal, and hence the side BE is equal to FE. But the perpendicular BE is given, and consequently FE is given both in position and magnitude; whence the point F is given, and therefore G the intersection of the straight line AF with CD.





COMPOSITION.

Let fall the perpendicular BE, and produce it equally on the opposite side, join AF meeting CD in G; AG and BG are the straight lines required.

For the triangles GBE and GFE, having the side BE equal to FE, GE common, and the contained angle BEG equal to FEG, are (I. 3. El.) equal; and consequently the angle BGE is equal to FGE or AGC.

When the points A and B lie on the same side of the line CD, the problem will always admit of a solution, since AGF must cross CD. If those points lie on opposite sides of CD and equidistant from it, AFG will become parallel to CD, and consequently the problem will fail unless they range in the same perpendicular, or the point F coincide with A, and the solution is indefinite, and every point in CD answers the conditions.

PROP. IV. PROB.

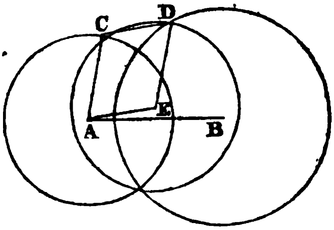
Between two given circles to place a given straight line with a given inclination.

Let it be required to place a given straight line CD at a given inclination to the line, joining the centres A and B of two given circles.

ANALYSIS.

Draw AE (I. 23. El.) parallel and equal to CD, and complete the rhomboid ACDE.

Because CD has a given inclination to AB, the angle BAE of its parallel is given, and consequently AE is given in position; but AE, being equal to CD, is given also in magnitude, and hence the point E is given. Again, ED, being equal to AC, is given in magnitude, and the point D lies in the circumference of a circle described from E. Wherefore D is given, and consequently CD, which is parallel to AE.



COMPOSITION.

Draw (I. 4. El.) AE at the given inclination, and equal to the given straight line, and from the centre E, with the radius of the circle of which A is the centre, describe another circle cutting the circle of which B is the centre in the point D, join ED, and parallel to it draw AC; CD being joined is the line required.

For AC and ED being by this construction made equal and parallel, CD (I. 27. El.) is likewise equal and parallel to AE, and this fulfils the conditions.

The point D will have two positions, unless the circle described from E touch the circle whose centre is B. If the points E and B coincide, and the circles become equal, the problem will evidently become indeterminate, or any parallel CD will fulfil the conditions.

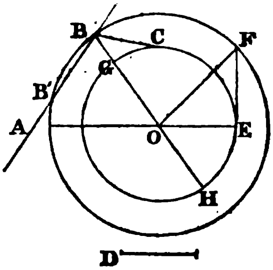
PROP. V. PROB.

From a straight line given in position to draw, equal to a given line, a tangent to a given circle.

Let it be required in the straight line AB to find a point B, from which the tangent BC, drawn to the circle having O for its centre, shall be equal to D.

ANALYSIS.

Through the centre O, draw BGOH, the square of BC is equivalent (III. 26. cor. 2. El.) to the rectangle BG, BH, or to (II. 17. El.) the excess of the square of OB above that of OG, whence the square of OB is equivalent to the squares of OC and BC, and is therefore given. Consequently, OB itself is given, and the point B, being the intersection of AB with a given circle, is likewise given.



COMPOSITION.

From O draw any radius OE, erect the perpendicular EF equal to D, join OF, and from O as a centre, describe through F a circle meeting AB in B, from which the tangent BC drawn to the interior circle is equal to D.

For (III. 26 cor. 2. El.) $BC^2 = GB.BH = OB^2 - OG^2$, or $OB^2 = BC^2 + OG^2$; but OF^2 or $OB^2 = OE^2 + EF^2$, and $BC^2 + OG^2 = OE^2 + OF^2$, or $BC^2 = EF^2$, and consequently BC is equal to EF or D.

PROP. VI. PROB.

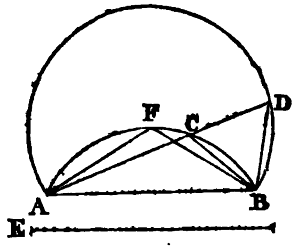
In a given arc of a circle, to inflect two chords that shall be together equal to a given straight line.

Let AFB be a given segment, it is required to inscribe the chords AC and BC, whose sum shall be equal to the line E.

ANALYSIS.

Produce AC till CD be equal to BC, and join BD.

The triangle BCD is evidently isosceles, and consequently (I. 30. El.) the exterior angle ACB is double of CDB; but ACB is given, and therefore its half, or ADB, is given. Whence ADB is contained in a given segment, and the inflected line AD, being equal to E, is given in position; wherefore the point C and the chords AC and BC are given.



COMPOSITION.

Bisect the arc AFB in the point F, from which centre, with the chord FA or FB as a radius, describe a circle, and inflect ACD equal to E; AC and BC being joined, are the chords required.

For join BD. The angle AFB at the centre (III. 15. El.) is double of ADB at the circumference, and consequently ACB is likewise double of ADB or CDB; but ACB is equal (I. 30. El.) to the two angles CDB and CBD, which are hence equal to the double of CDB, and therefore CDB and CBD are mutually equal, and contain an isosceles triangle. Whence BC is equal to CD, and AC and BC together are equal to AD or E.

It is evident that E, the sum of the inflected chords, can never exceed the diameter of the exterior circle, or the amount of the equal chords AF and BF. In other cases, the point D, and consequently C will have two positions.

Cor. Hence if a segment ADB of a circle be constituted on the same base with another, but containing only half the vertical angle, any chord ACD will have its intercepted portion CD equal to the chord BC drawn from the other side of the interior segment.

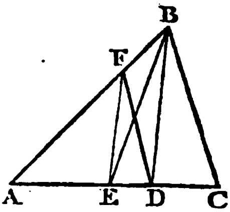
PROP. VII. PROB.

To bisect a given triangle, by a straight line drawn from a given point in one of its sides.

Let it be required, from the point D, to draw DF, bisecting the triangle ABC.

ANALYSIS.

If D be the middle of AC, the line DB drawn to the vertex will obviously (II. 2. El.) divide the triangle into two equal portions. But, if not, bisect (I. 7. El.) the side AC in E, and join EB, EF and BD. The triangle ABE is equal to EBC, and is consequently the half of ABC, wherefore ABE is equal to AFD, and, taking AFE from both, the remaining triangle EFB is equal to EFD; and since these triangles stand on the same base, they must (II. 3. El.) have the same altitude, or EF is parallel to BD. But the points B and D being given, the straight line BD is given in position, and consequently EF is also given in position.



COMPOSITION.

Having bisected AC in E and joined BD, draw EF parallel to it, meeting AB in F; the straight line DF divides the triangle ABC into two equal portions.

For join BE. Because BD is parallel to EF, the triangle EFB (II. 1. El.) is equal to EFD; and, adding AFE to each, the triangle AFD is equal to ABE, that is, to the half of the triangle ABC.

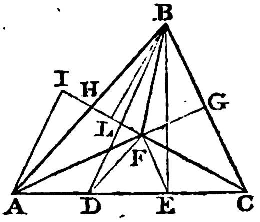
PROP. VIII. PROB.

To find a point within a given triangle, from which straight lines drawn to the several corners will divide the triangle into three equal portions.

Let F be the required point, from which the lines FA, FB, and FC trisect the triangle ABC.

ANALYSIS.

Draw FD, FE parallel to the sides BA, BC, and join BD, BE. Since FD is parallel to AB, the triangle ABF (II. 1. El.) is equal to ABD, which is hence the third part of ABC; and, for the same reason, the triangle BFC is equal to BEC, which is also the third part of ABC. Wherefore the bases AD and EC are each the third part of AC, and consequently the points of section D and E are given; hence (I. 23. El.) the parallels DF and EF are given in position, and their point of concourse is therefore given.



But the point F may be otherwise investigated. For produce AF and CF to G and H. The triangle DFE is evidently (I. 22. El.) similar to ABC, and therefore AC : AB : : DE : DF, but AC=3DE, and consequently (V. 8: and 5. El.) AB=3DF. Again, because AH and DF are parallel, AC : AH : : DC : DF, and (V. 13. El.) 2AC : 2AH : : 3DC : 3DF; but 2AC=6AD=3DC, and 2AH =3DF=AB. Hence AB is bisected in H; and, for the same reason, BC is bisected in G. Wherefore the points H and G being thus given, the intersection F of the straight lines CH and AG is likewise given.

COMPOSITION.

Bisect AB and BC (I. 7. El.) in H and G, join CH and AG, and, from their point of intersection, draw FA, FB, and FC; the triangle ABC will thus be divided into three equal portions.

For, from the points A and B let fall the perpendiculars AI and BL. The triangles HAI and HBL, having the angles AHI and AIH equal to BHL and BLH, and the side AH equal to BH, are (I. 20. El.) equal, and consequently AI=BL. The triangles AFC and BFC, standing on the same base CF, and having equal altitudes AI and BL, are equal (II. 2. El.). And, in the same manner, it is shown that the triangles AFC and AFB are equal. Wherefore the whole triangle ABC is divided into three equal triangles, having their common vertex at the point F.

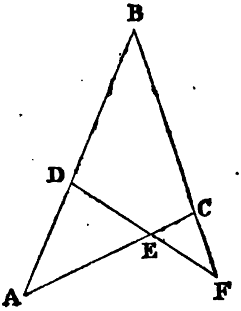
PROP. IX. PROB.

From a given point in the side of a given triangle, to draw a straight line to the opposite side, such that the rectangle under its segments intercepted by the base shall be equivalent to the rectangle under the segments of the base itself.

Let it be required from D in the side AB of the triangle ABC, to draw DF to the opposite side, so that the rectangle DE, EF shall be equivalent to the rectangle AE, EC.

ANALYSIS.

Since DE.EF=AE EC, it follows that (V. 6. El.) DE : AE : : EC : EF ; but the vertical angles AED and CEF are equal, and consequently (VI. 13. El.) the triangles EAD and EFC are similar. Wherefore the angle ADE or ADF, being equal to ECF, is given, and hence DEF is given in position.



COMPOSITION.

From D (I. 4. El.) draw DEF, at an angle ADF equal to ACF, and the problem is solved. For the triangles EAD and EFC, having their vertical angles equal, and the angles ADE and ECF by construction equal, are (VI. 11. El.) similar. Whence DE : AE : : EC : EF, and, therefore, (V. 6. El.) DE.EF=AE.EC.

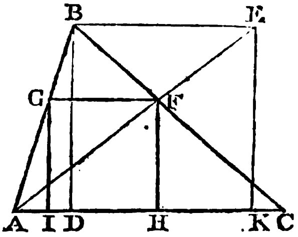
PROP. X. PROB.

To inscribe a square in a given triangle.

Let ABC be the triangle in which it is required to inscribe a square IGFH.

ANALYSIS.

Join AF, and produce it to meet a parallel to AC in E, and let fall the perpendiculars BD and EK.



Because EB is parallel to FG or AC, AF : AE : : FG : EB (VI. 2. El.) ; and since the perpendicular EK is parallel to FH, AF : AE : : FH : EK. Wherefore FG : EB : : FH : EK; but FG=FH, and consequently (V. 8. and 5. El.) EB=EK. Again, EK, being equal to BD, the altitude of the triangle ABC is given, and, therefore, EB is given both in position and magnitude; whence the point E is given, and the intersection of AE with BC is given, and consequently the parallel FG and the perpendicular FH are given, and thence the square IGFH.

COMPOSITION.

From B draw BD perpendicular and BE parallel to AC, make BE equal to BD, join AE, intersecting BC in F, and complete the rectangle IGFH.

Because BE and EK are parallel to GF and FH, AE : AF : : BE : GF, and AE : AF : : EK : FH; wherefore BE : GF : : EK : FH; but BE=EK, and consequently GF=FH. It is hence evident that IGFH is a square.

PROP. XI. PROB.

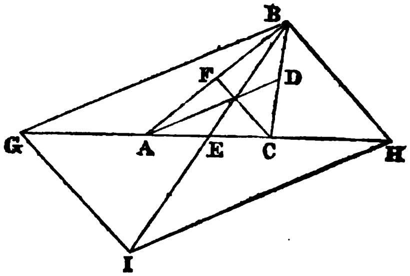
To construct a triangle, of which the three straight lines drawn from the angular points to bisect the opposite sides, are given.

Let the straight lines AD, BE, and CF, bisecting the opposite sides of the triangle ABC, be given; to construct that triangle.

ANALYSIS.

Through the vertex B draw BG and BH parallel to AD and CF, and meeting the base produced; extend BE to an equal distance beyond E, and join GI and HI.

From the property of parallel and diverging lines, it is obvious that BG is the double of AD, and BH the double of CF; for the same reason, DA bisects CG, and FC bisects AH, or AC is equal both to GA and to CH, and consequently GE is equal to EH. Wherefore the triangles GEI and BEH, having their vertical angles equal, likewise the containing sides GE, EI, equal to EH, EB are (I. 3. El.) equal. Hence the base GI is equal to BH or the double of CF. All the sides of the triangle BGI are thus given, and consequently E, the middle point of BI, is given. But AE or EC, being the third part of GE, is given; and the two points A and C, with the vertex B, determine the triangle.



COMPOSITION.

With the doubles of the given straight lines for its sides, construct (I. 1. El.) the triangle BGI, and complete the rhomboid BGIH: Draw the diagonal GH, which trisect (I. 36. El.) in the points A and C, and join BA and BC. Then ABC is the triangle required.

For join BI, which must consequently bisect GH in E. Since AD bisects both CG and CB, the line AD (VI. 2. El.) must be parallel to BG, and equal to half of it. In the same way it is proved that CF is the half of BH; and BE is evidently the half of the diagonal BI; so that the conditions of the problem are fulfilled.

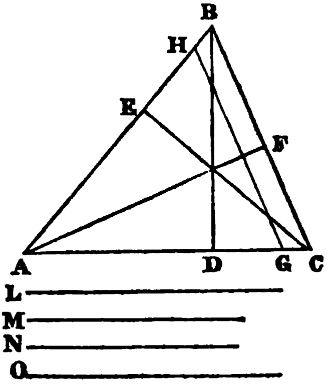
PROP. XII. PROB.

Given the perpendiculars from the angles of a triangle on the opposite sides, to construct the triangle.

Let ABC be a triangle of which the perpendiculars drawn from A, B and C, the opposite angles, are expressed by L, M and N, to determine the triangle.

ANALYSIS.

Since the area of a triangle is equivalent to half the rectangle under any side, and the perpendicular let fall on it, (II. 5. El.) the sides themselves must be reciprocally as the several perpendiculars. Wherefore the ratio of AB to BC, being the same as that of AF to CE, is given : again, the ratio of BC to AC, or that of BD to AF, is given; consequently the mutual ratios of the sides of the triangle ABC being given, the triangle itself is given in species. But it is likewise given in magnitude; for the right angled triangle ABD having its acute angles and its perpendicular BD given, is evidently given; and the hypotenuse AB being hence given, the other two sides of the triangle ABC, which have a given ratio to it, are hence also given.



COMPOSITION.

Make M : N : : L : O, draw AG equal to O, and on this base, construct a triangle having the sides AH and GH equal to L and N; produce AH till it meet a line at the distance M from AC in the point B, draw BC parallel to HG, and ABC is the triangle required.

For, from the properties of parallel and diverging lines, AH is to AG as AB to AC, or (II. 5. and V. 25. cor. 3. El.) as the perpendicular BD to CE; that is, L : O : : BD : CE. For, by construction, M : N : : L : O; and, therefore, M : N : : BD : CE; but BD was made equal to M, and consequently (V. 4. El.) CE must be equal to N. Again, because HG : AG : : BC : AC : : BD : AF, or N : O : : BD : AF, therefore M : L : : BD : AF; but BD = M, whence AF = L. The triangle ABC thus fulfils the conditions, having all its perpendiculars AF, BD, and CE equal respectively to L, M and N.

PROP. XIII. PROB.

To describe a triangle whose sides are given in magnitude, and pass through three given points.

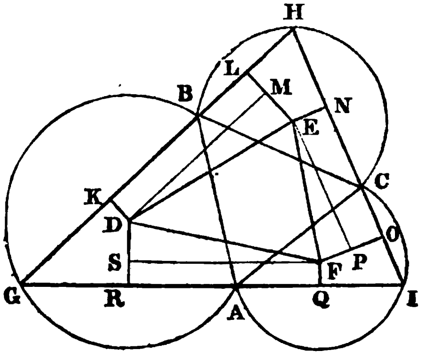
Let it be required to construct a triangle, of which the sides GI, GH and HI are given, and moreover pass through the given points A, B and C.

ANALYSIS.

Join AB, BC and AC. Since the sides of the triangle GHI are given, its angles are likewise given. The given angle AGB, standing on a given base AB, is therefore contained in a given segment of a circle, of which the centre is D. For the same reason, the angles BHC and AIC are contained in given segments of circles, having their centres in E and F. Let fall the perpendiculars DK, EL, EN, FO, FQ and DR, and draw the parallel DM, EP and FS. Those perpendiculars (III. 4. El.) bisect the chords BG, BH, CH, CI, AI and AG; and BK and BL, being the halves of BG and BH, KL or DM is the half of the side GH, and is consequently given. But the right angle DME is contained in a given semicircle, and therefore DM and the parallel GH are given in position. In like manner, it is shown, that HI and GI are given in position; whence the triangle GHI is given both in position and magnitude.

COMPOSITION.

Join the points A, B and C; and having formed a triangle having sides equal to the given lines GI, GH and HI, on AB, BC and AC construct (III. 23. El.) arcs containing the angles G, H and I, and whose centres are D, E and F; on DE, EF and DF describe semicircles, in which inflect DM, EP and FS respectively equal to the halves of GH, HI and GH; and through the points B, C and A, but parallel to DM, EP and FS, draw GH, HI and GI, which will compose the triangle required.



For having joined EM, FP and DS, produce them to GH, HI and GI, and draw the parallels DK, EN and FQ.

It is evident (III. 4. El.) that GH or BG and BH is equal to twice BK, and twice BL, twice KL; whence GH is double of DM. And, in the same manner, it is shown that HI is the double of EP, and GI the double of FS. The triangle GHI is therefore contained by the given sides, and it passes through the given points A, B and C.

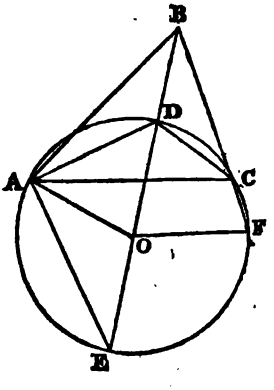
PROP. XIV. PROB.

Given two sides of a triangle, and the distance of their vertex from the centre of the inscribed circle, to construct the triangle.

Let the sides AB and CB, together with the straight line BD drawn from the vertex B to D the centre of the inscribed circle be given, to construct the triangle ABC.

ANALYSIS.

Join AD and DC, about the triangle ADC (III. 9. cor. El.) describe a circle, produce BD to E, and join AE.



Because D is the centre of the inscribed circle, the angle ABE (IV. 10. El.) is equal to CBD; but AEB or AED (III. 15. El.) is equal to ACD, that is, (IV. 10. El.) to DCB. Wherefore the triangle BAE is similar to BDC, and BE : BA :: BC : BD; whence the rectangle under BE and BD, being (V. 6. El.) equivalent to the rectangle under AB and BC, is given; but BD being given, BE and the point E are likewise given. Again, the angle DAE consists of DAC and CAE or (III. 15. El.) CDE; of which, DAC is half the angle BAC, (IV. 10. El.) and CDE is (I. 30. El.) equal to the angles CBD and DCB, or to half the angles ABC and ACB; consequently the angle DAE is half of all the angles of the triangle ABC, and is therefore a right angle, and (III. 19. El.) stands on the diameter DE. The circle CDAE is hence given, with the points E, D and B, and therefore BA and BC being given, the points A and C are likewise given, and thence the triangle ABC.

COMPOSITION.

Produce BD to E, so that BD : BC : : BA : BE; on DE as a diameter describe a circle, and from B inflect to the circumference lines equal to BA and BC; then, AC being joined, ABC is the triangle required.

For produce BC to meet the circumference again in F, and join OA, OF. From the property of a circle, the rectangle BF, BC is equivalent to BD, BE; but, by construction, this rectangle BD, BE is equivalent to BA, BC, which is therefore equivalent to BF, BC, and consequently BA is equal to BF. Wherefore the triangle ABO has all its sides respectively equal to those of EBO, and these triangles are hence equiangular. The line BD consequently bisects the angle ABC; but since, by construction, BD : BC : : BA : BE, and the angle ABE is equal to CBD, the triangles BAE and BDC are (VI. 13. El.) similar; whence the angle AED, which (III. 15. El.) is equal to ACD, is likewise equal to DCB, and CD bisects the angle ACB. The point D is, therefore, by its position the centre of a circle inscribed within the triangle ABC.

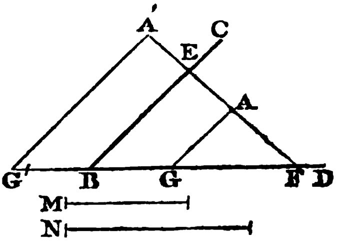
PROP. XV. PROB.

To draw a straight line through a given point, so that its portions, terminated by two straight lines given in position, shall have a given ratio.

Let A be a given point, and BC, BD two straight lines given in position; it is required to draw EAF, such that EA shall be to AF as M to N.

ANALYSIS.

Draw AG parallel to BC and meeting BD in the point G, which is thus given. The diverging lines FE, FB are cut proportionally by the parallels BE, GA, (VI. 1. El.), and consequently EA : AF : : BG : GF; but the ratio of EA to AF is given, and therefore likewise the ratio of BG to GF; and BG being given, GF is given, and the point F; whence the straight line EAF is given in position.



COMPOSITION.

Draw AG parallel to BC, make (VI. 3. El.) BG : GF : : M : N, and join FAE; this is the line required.

For, BE and AG being parallel, (VI. 1. El.) EA : AF : : BG : GF; but by construction BG : GF : : M : N, and therefore EA : AF : : M : N.

PROP. XVI. PROB.

From the vertex of a given triangle, to draw a straight line which shall be a mean proportional between the segments of the base.

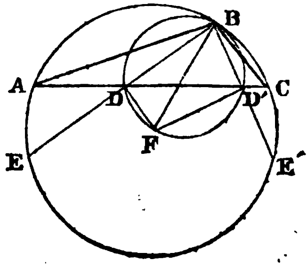
CASE I.

When the section is internal.

Let ABC be the given triangle, from the vertex of which it is required to draw BD a mean proportional to the segments AD and DC of the base.

ANALYSIS.

Describe a circle about the triangle (III. 9. cor. El.), and produce BD to meet the circumference in E. Since BD is a mean proportional between AD and DC, its square is (V. 6. El.) equivalent to the rectangle under AD and DC; but this rectangle is equivalent (III. 26. El.) to the rectangle under BD and DE, which is therefore equivalent to the square of BD, and consequently BD itself is equivalent to DE. Wherefore (III. 5. El.) FD being joined, is perpendicular to BE, and the right angle FDB (III. 19. El.) is contained in a semicircle, whose intersection with the base of the triangle determines the position of the point D.



COMPOSITION.

About the triangle ABC (III. 9. cor. El.) describe a circle, join its centre F with the vertex B, and on this straight line, as a diameter, describe another circle cutting the base of the triangle in the point D. Join BD, and it is the line required.

For produce BD to the circumference, and join FD.

The angle FDB (III. 19. El.) is a right angle, and consequently BD is equal to DE. But the rectangle AD, DC (III. 26. El) is equivalent to BD, DE, or to the square of BD. Whence BD is a mean proportional between AD and DC, the segments of the base.

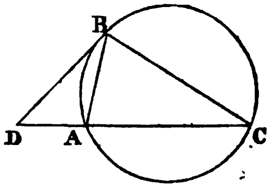
CASE II.

When the section is external.

Let it be required from the vertex B to draw to a point beyond the base the straight line BD, a mean proportional to the segments DA and DC.

ANALYSIS.

Describe a circle about the given triangle. And since, by hypothesis, the square of DB is equivalent to the rectangle under DA and DC, DB (III. 26. cor. 2. El.) must touch the circumference. But this tangent BD is given in position, consequently is likewise its intersection D with the produced base.



COMPOSITION.

Having described a circle about the triangle, the tangent BD applied at the vertex B will be a mean proportional between DA and DC; for (III. 26. cor. 2. El.) the square of BD is equivalent to the rectangle DA, DC.

SCHOLIUM.

In the first case, if the vertical angle be acute, the centre of the circumscribing circle will lie within the triangle, and consequently the interior circle not intersecting the base, the construction fails, and the problem becomes then impossible. When the vertical angle is right, the centre F (III. 19. El.) will bisect the base, and the interior circle will pass through that point, and again cut the base at D, forming the perpendicular chord BD. But when the vertical angle is obtuse, the centre F will always fall below the base, and the interior circle will assign two different positions BD and BD′ of the mean proportional. In the second case, the line BD can have only a single position; but when the triangle becomes isosceles, the problem will fail.

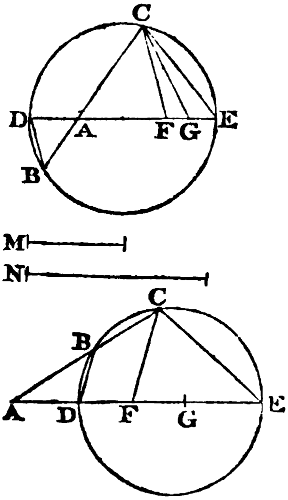
PROP. XVII. PROB.

Through a given point, to draw a straight line that shall be cut in a given ratio, by the circumference of a given circle.

Let A be the given point, and BDCE the given circle; it is required to draw BC, so that BA shall be to AC as the line M to N.

ANALYSIS.

Draw the diameter DAE, join DB, CE, and draw CF parallel to DB. Because the point A and the centre of the circle are given, the diameter DE is given in position, and consequently its extremities D and E are given. But, DB and CF being parallel, BA : AC : : DA : AF (VI. 1. El.), wherefore the ratio of DA to AF is given, and since DA is given, AF is also given. Again, BA.AC = AD.AE (III 26. El.), and consequently AE : AC :: BA : DA; but BA : DA : : AC : AF (VI. 1. El.), whence AE : AC : : AC : AF; or AC is a mean proportional between AF and AE, and is, therefore, given. The point C is thus given, and consequently BC.



COMPOSITION.

Having drawn the diameter DE, make DA: AF::M: N, find (VI. 16. El.) AG a mean proportional between AF and AE, and inflect AC equal to it ; BAC is the straight line required.

For join DB, CF, and CE. Since (III. 26. El.) the rectangle BA, AC is equivalent to the rectangle DA, AE, it follows that AE : AC : : BA : DA ; but, by construction, AE : AC : : AC : AF, and therefore AC : AF : : BA : DA ; hence (VI. 1. cor. 1. El.) CF is parallel to DB, and consequently, BA is to AC, as DA to AF, that is, as M to N.

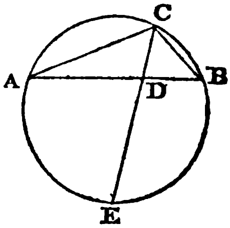
PROP. XVIII. PROB.

From two given points in the circumference of a given circle, to inflect, to another point in that circumference, straight lines which shall have a given ratio.

From the points A and B, let it be required to inflect AC and BC in a given ratio.

ANALYSIS.

Draw (I. 5. El.) CE bisecting the vertical angle ACB. Therefore (VI. 10. El.) AC : CB : : AD : DB, and consequently the ratio of AD to DB is given, and thence (VI. 4. El.) the point D is given. But since the angle ACE is equal to BCE, the arc AE is (III. 16. cor. El.) equal to the arc EB, and therefore the point E is given. Whence the points E and D being given, the straight line EDC is given in position, and consequently the point C and the chords AC and BC are given.



COMPOSITION.

Bisect (III. 13. El.) the arc AEB in E, divide AB (VI. 4. El.) in the given ratio at D, join ED, and produce it to meet the opposite circumference in C; the chords AC and CB are in the given ratio.

For since the arc AE is equal to BE, the angle ACD is (III. 16. cor. El.) equal to BCD, and consequently (VI. 10. El.) AC : CB : : AD : DB, that is, in the given ratio.

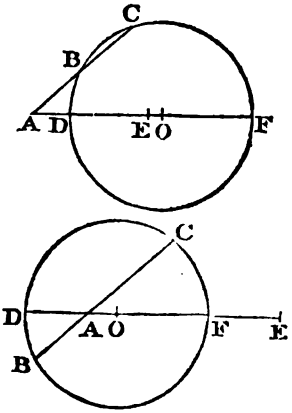
PROP. XIX. PROB.

Through a given point, to draw a straight line to a circle, so that the rectangle under the part limited by the circumference and the segment included within the circle, shall be equal to a given space.

Let it be required, through the point A, to draw ABC, such that the rectangle AB, BC, shall be equal to a given space.

ANALYSIS.

Through the centre O draw AF, and (II. 8. El.) find AE, which forms with AD a rectangle equal to the given space. Because (III. 26. El). AB.AC = AD.AF, and, by construction, AB.BC = AD.AE; it follows (V. 6. El.) that AD : AB : : AC : AF : : BC : AE; whence (V. 19. cor. 1. El.) AD : AB : : AC ∽ BC, that is AB : AF ∽ AE or EF. Wherefore AB is a mean proportional between AD and EF; but AE being given, EF is also given, and consequently AB is given both in magnitude and position.



COMPOSITION.

Draw AF through the centre of the circle, make (II. 8. El.) the rectangle AD, AE equal to the given space, find (VI. 16. El.) a mean proportional to AD and EF, and inflect this from A towards B; the rectangle AB, BC is equal to the given space.

For (V. 6.) AD : AB : : AB : EF, and (V. 6. and III. 26. El.) AD : AB : : AC : AF, whence (V. 19. cor. 1. El.) AD : AB : : $AC \mp AB$ or BC : $AF \mp EF$ or AE, and consequently $AD.AE = AB.BC$.

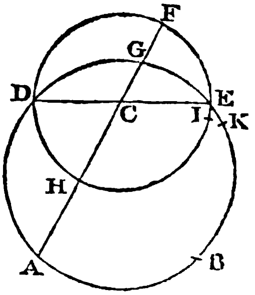
PROP. XX. PROB.

Through two given points, to describe a circle bisecting the circumference of a given circle.

Let A and B be two points, through which it is required to describe a circle ADGEB, that shall bisect the circumference of the circle HDFE.

ANALYSIS.

Join D, E, the points of intersection. Because DFE is, by hypothesis, a semicircumference, DE is a diameter, and must, therefore, pass through the centre C. Join AC, and produce it to F. Since $DC = CE$, it is evident (III. 26. El.) that $AC.CG = DC^2 = HC.CF$; but the rectangle HC, CF is given, and consequently the rectangle AC, CG is also given; and AC being given,



CG is hence given, and the point G. Wherefore the three points A, G, and B being given, the circle AGB is (III. 9. El.) given.

COMPOSITION.

Through C, the centre of the given circle, draw ACF, make (VI. 3. El.) AC : HC : : CF or HC : CG, and through the three points A, G, and B, describe (III. 9. cor. El.) the circle AGB: This will bisect the circumference HDFE.

For, through one of the points of intersection, draw the diameter DCI, and produce it to meet the circumference of the circle AGB in K. Because AC : HC : : HC : CG, the square of HC is (V. 6. El.) equal to the rectangle AC and CG; but (III. 26. El.) $HC^2 = DC.CI$, and $AC.CG = DC.CK$: wherefore $DC.CI = DC.CK$, and (II. 3. cor. El.) $CI = CK$, or the points I and K are one, and the circle AGB passes through both extremities of the diameter of HDFE.

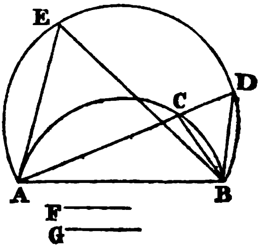
PROP. XXI. PROB.

Two arcs of different circles being constituted on the same chord, to draw from its extremity a straight line, of which the portion intercepted by those arcs shall be equal to a given line.

Let the arcs ACB and AEB stand on the same base AB; it is required to draw ACD, such that CD shall be equal to F.

ANALYSIS.

Join BC and BD. The angle CDB contained in the arc AEB is given; and so is ACB, and consequently its supplement BCD; wherefore all the angles of the triangle CBD are given, and the triangle itself is given in species. But it is likewise given in magnitude, for the side CD is given. Hence BD is given, and therefore the point D and the line ACD are also given.



COMPOSITION.

Draw the tangent AE and join BE; make AE to BE as F to G, and in the exterior segment inflect G from B to D; ACD being drawn in the line required.

For join BC and BD, the angle AEB (III. 16.) is equal to ADB or CDB, and the angle EAB is equal to the angle in the segment alternate to ACB, or to the exterior angle BCD; consequently the triangle AEB is similar to CDB, and AE : BE :: CD : BD; but, by construction, AE : BE : : F : G, and hence CD : BD: : F : G; wherefore (V. 4. El.) since BD was made equal to G, CD must be equal to F.

PROP. XXII. PROB.

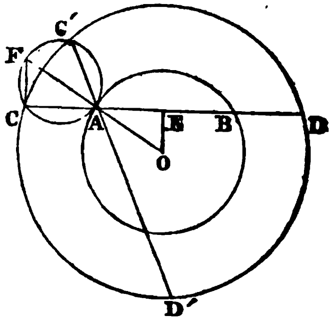
Through a given point in the interior circumference of two given concentric circles, to draw a straight line which shall intercept chords in a given ratio.

Let it be required, through the point A, to draw the straight line CAD, such that the chords AB and CD of the concentric circles shall have a given ratio.

ANALYSIS.

From the common centre O let fall the perpendicular OE, and draw CF parallel to this, meeting OA produced in F.

The perpendicular OE (III. 4. El.) bisects both the chords AB and CD; wherefore the ratio of AE to CE, being thus the same as that of AB to CD, is given; but (VI. 1. El.) the ratio is likewise the same as that of OA to OF, which is consequently given; whence OF and the point F are given. Now ACF being a right angle, is contained in a semicircle described on the given diameter AF, and therefore its intersection C is given.



COMPOSITION.

Join the common centre O with the point A, and produce the line, till OA be to OF in the given ratio; on AF as a diameter describe the circle ACF, cutting the exterior circle in C or C′; the compound chord CAD or C′AD′ is the line required.

For, let fall on CD the perpendicular OE. The angle ACF being (III. 19. El.) a right angle, it is evident that OE is parallel to CF, and consequently that OA is to OF as AE to CE, or as their doubles AB to CD, which are chords therefore drawn from the point A in the given ratio.

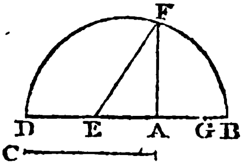
The compound chord has obviously two positions, except when AF is equal to AC. But if AF be less than AF, the problem is impossible.

PROP. XXIII. PROB.

To cut a given straight line, such that the square of one part shall be equivalent to the rectangle under the remainder and another given straight line.

Let AB be a straight line, from which it is required to cut off a segment AG, whose square shall be equivalent to the rectangle under the remainder and the straight line C.

ANALYSIS.



Produce BA till AD be equal to C, on DB describe a semicircle and erect the perpendicular AF. Because $AG^2 = C \times GB$, it follows (V. 6. El.) that $DA : AG :: AG : GB$; wherefore (V. 19. El.) $DA : AG :: DG : AB$, and consequently $DA.AB = AG.DG$; but (III. 26. cor. 1. El.) $DA.AB = AF^2$; and therefore $AG.DG = AF^2$; whence AF is equal to a tangent drawn from G to a semicircle described on DA. Bisect DA in E, and join EF; and because $AG.DG = AF^2$, add EA^2 to each, and $AG.DG + EA^2$, or (II. 17. cor. 2. El.) EG^2, is equivalent to AF^2+EA^2 or (II. 10.) EF^2; whence EG is equal to EF, and is therefore given.

COMPOSITION.

Having produced AD equal to C, and described on BD a semicircle, erect the perpendicular AF, bisect AD in E, join EF and make EG equal to it; the square of the segment AG thus formed in AB is equivalent to the rectangle under the remaining part GB and the given line C.

For EFA being a right angled triangle $EF^2=EA^2+AF^2$ (II. 10. El.), and consequently $AF^2=EF^2-EA^2$, or EG^2-EA^2; and since (II. 17. El.) $EG^2-EA^2=(EG+EA)(EG-EA)$, or DG.AG, therefore $AF^2=DG.AG$. But (III. 26. cor. 1. El.) $AF^2=DA.AB$; whence $DG.AG=DA.AB$, and $AG:AB::DA:DG$ (VI. 6. El.); wherefore (V. 11. and V. 7. El.) $AB-AG$, or $GB:AG::DG-DA$ or $AG:DA$, whence (V. 6. El.) $AG^2=GB.DA$.

Cor. If DA, or C, be equal to AB, then $AG^2=AB.BG$, or $AB:AG::AG:BG$, and, therefore, the line AB is now divided in extreme and mean ratio, at the point G. The construction also becomes evidently the same with that which was given in Book II. Prop. 19. of the Elements, for the medial section of a line, and which is really a simple case of the same problem.

PROP. XXIV. PROB.

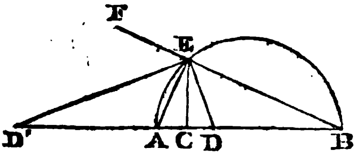
To divide a straight line, such that its segments shall have the subduplicate ratio of those formed by another given section of it.

Let it be required to divide the straight line AB in D, such that the segments AD, DB shall be in the subduplicate ratio of other segments AC, CB.

1. *Let the given section be internal.*

ANALYSIS.

On AB describe a semicircle, erect the perpendicular CE, and join AE, BE and ED or ED′. Because (III. 19. El.) AEB is a right angle, the ratio of AE to BE (VI. 15. cor. 1. El.) is the subduplicate of that of AC to BC, and consequently AE : BE : : AD : BD, or AD′ : BD′; wherefore (VI. 10. cor. El.) the vertical angle AEB is bisected internally or externally by ED or ED′. But the perpendicular and the semicircle being both given, the vertex E, the straight line ED or ED′, and the point of section D or D′, are likewise given.



COMPOSITION.

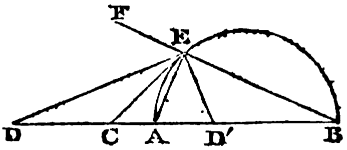
Having on AB described a semicircle, erect the perpendicular CE, join EA, EB, and draw ED or ED′ bisecting the angle AEB or its adjacent angle AEF; the internal segments AD, DB, or the external segments AD′, D′B, are in the subduplicate ratio of AC to CB.

For (VI. 10. El.) AE : BE : : AD : DB, or AD′ : D′B; but the triangle AEB being right-angled, AE is to BE (VI. 15. cor. El.) in the subduplicate ratio of AC to CB, and consequently AD is to BD, or AD′ to D′B, in the same subduplicate ratio.

2. *Let the given section be external.*

ANALYSIS.

On AB describe a semicircle, draw the tangent CE, and join AE, BE, and ED or ED′. The triangles ACE and ECB are similar, for (III. 21. El.) the angle CEA is equal to CBE in the alternate segment, and BCE is common to both triangles; whence AC : CE : : CE : BC, and consequently (V. def. 20. El.) the ratio of CE to BC is the subduplicate of that of AC to BC. But in these similar triangles, AE : CE : : BE : BC, and alternately AE : BE :: CE : BC; wherefore AE : BE : : AD : DB, or AD′ : D′B, and the vertical angle AEB (VI. 10. cor. El.) is bisected externally or internally by ED or ED′.



COMPOSITION.

Having described a semicircle on AB, apply (III. 22. El.) the tangent CE, join AE, BE, and draw ED or ED′ bisecting externally or internally the vertical angle AEB; the external segments AD, DB, or the internal segments AD′, D′B are in the subduplicate ratio of AC to BC.

For the angle CEA being (III. 21. El.) equal to CBE, and BCE common to the two triangles ACE and ECB, these are similar, and AC : CE : : CE : BC; whence the ratio of CE to BC is the subduplicate of that of AC to BC. Again, from the same similar triangles, AE : CE :: BE : BC, or alternately AE : BE : : CE : BC, and therefore AE is to BE in the subduplicate ratio of AC to BC. But (VI. 10. El.) AE : BE : : AD : DB, or AD′ : D′B, and consequently the ratio of AD to DB or of AD′ to D′B is the subduplicate of that of AC to BC.

Cor. In the second case, the angle CD′E (I. 30. El.) being equal to D′EB and D′BE, which are equal to D′EA and AEC, is therefore equal to CED′, and the triangle D′CE is hence isosceles. Again, the angle DEF, equal by hypothesis to DEA or CED and AEC, is (I. 30. El.) equal to CDE and DBE or AEC, and consequently the triangle DCE is likewise isosceles. Wherefore CE=CD=CD′, and thus, without bisecting the vertical angle, the point D or D′ is found from the tangent CE, which is a mean proportional between the segments AC and BC.

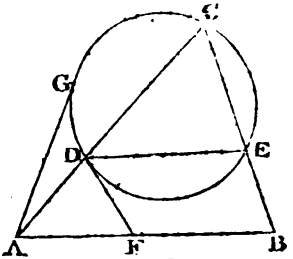
PROP. XXV. PROB.

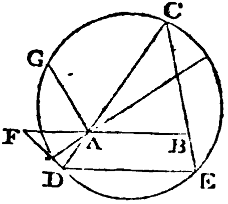
Through two given points, to draw straight lines to a point in the circumference of a given circle, so that the chord of the intercepted segment shall be parallel to the straight line which joins the given points.

Let it be required, from the points A and B, to inflect AC and BC cutting the given circumference in D and E, such that the chord DE shall be parallel to AB.

ANALYSIS.

From D (III. 20. cor. El.) draw the tangent DF, meeting AB in F. The angle FDE is equal (III. 21. El.) to the angle ECD, or its supplement, in the alternate seg-





ment; but DE being parallel to AB, the angle FDE or its supplement is (I. 22. El.) equal to the alternate angle AFD, which is consequently equal to the angle ECD or ACB; wherefore the triangles ADF and ABC, having likewise a common angle CAB, are (VI. 11. El.) similar, and AD : AF : : AB : AC, and hence AD.AC = AF.AB. But since the point A and the circle DCE are given, the rectangle AD, AC is also given; for it is equal to the square of the tangent AG (III. 26. cor. 2. El.), when A lies without the circumference,—and equal to the square of AG (III. 26. cor. 1. El.) a perpendicular to the diameter, in the case where that point lies within the circle. Hence the rectangle AF, AB is given; and AB being given, AF is likewise given, and consequently the point F. Wherefore the tangent FD is given in position; and since the point A is given, the straight line AC is given, and thence BC and the intersection E.

COMPOSITION.

If the point A be without the circle, draw the tangent AG; or if it lie within the circle, erect AG perpendicular to the diameter which passes through it. Make (VI. 3. El.) AB : AG : : AG : AF, from F draw the tangent FD, join AD, and produce it to meet the opposite circumference in

C, join CB, cutting the circle in E; the straight line DE is parallel to AB.

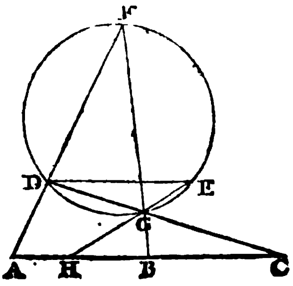
For, since AB : AG :: AG : AF, $AG^2 = AB.AF$; but (III. 26. cor. 1. and 2. El.) $AG^2 = CA.AD$, whence $AB.AF = CA.AD$, and consequently (V. 6. El.) AB : AC :: AD : AF. Wherefore (VI. 13. El.) the triangles BAC and DAF, having the sides about their common angle proportional, are similar, and hence the angle ACB is equal to AFD; but (III. 21. El.) ACB or DCE is equal to EDF or its supplement, and consequently the angle AFD is equal to EDF or its supplement, and (I. 22. cor. El.) the chord DE is parallel to AB.

PROP. XXVI. PROB.

From two given points, to inflect straight lines to the circumference of a circle, such that the chord of their intercepted arc shall tend to a given point in the direction of the former.

Let it be required, from the points A and B, to inflect AF and BF, so that the chord DG produced shall meet the extension of AB in the point C.

ANALYSIS.



Draw DE parallel to AC, join EG, and produce it to meet AB in H.

The angle BHG is equal to the alternate angle GED, which is equal (III. 15. El.) to GFD, and consequently the angles BHG and BFA are equal, and the triangles BGH and BAF are similar. Wherefore BG : BH : : BA : BF, and BG.BF = BH.BA; but the rectangle BG, BF is given, since it is equal to the square of a tangent drawn from B, and hence BH.BA is given, and the point H. The problem is thus reduced to the last Proposition, and only requires, from the points C and H, to inflect CD and HE, such that DE, the chord of their intercepted arc, may be parallel to HC.

COMPOSITION.

From the point B draw a tangent to the circle, find BH a third proportional to BA and it, and, by the last Proposition, inflect HE and CD such that DE shall be parallel to HC; then BG, being produced to F in the circumference, ADF forms one straight line.

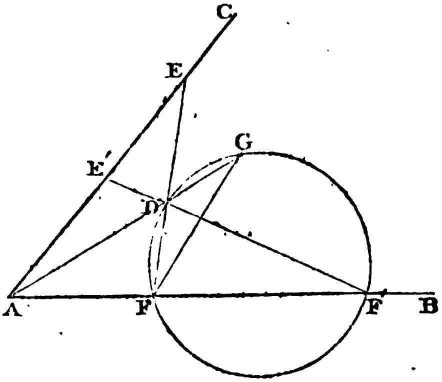
For since the tangent is a mean proportional to BA and BH, the rectangle BA, BH will be equivalent to the square of that tangent, or (III. 26. cor. 2. El.) to the rectangle BG, BF; consequently (V. 6.) BA : BF : : BG : BH, and (VI. 13. El.) the triangles BAF and BGH are similar; wherefore the angle BFA is equal to BHG, which (I. 22. El.) is equal to GED, and this again (III. 16. El.) is equal to GFD; whence BFA is equal to GFD, or the straight lines FA and FD lie in the same direction from F.

PROP. XXVII. PROB.

Through a given point to draw a straight line, so that the rectangle under its segments, intercepted by two straight lines given in position, shall be equal to a given space.

Let AB, AC be two straight lines, and D a point through which it is required to draw EF, such that the rectangle under its segments ED, DF shall be equal to a given space.

ANALYSIS.

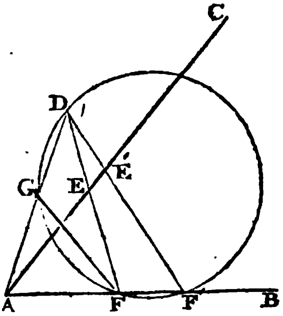


Join AD, from F draw (I. 4. El.) FG, making an angle DFG equal to DAE, and meeting AD or its production in G. The triangles ADE and FDG, being thus evidently similar, AD : ED : : DF : DG, and consequently (V. 6. El.) AD.DG = ED.DF. But the rectangle ED, DF is given, and therefore also the rectangle AD, DG; and since AD is given in position and magnitude, DG and the point G are given. Again, the angle DFG, being equal to DAC, is given, and thence (III. 23. El.) the segment of the circle which contains it; wherefore the contact or intersection of that arc with the straight line AB is given, and consequently the position of EF or E′F′ is likewise given.

COMPOSITION.

Join AD, make the rectangle AD, DG equal to the given space, and on DG describe (III. 27. El.) an arc containing an angle equal to DAC, and meeting AB in F or F′; EDF or EDF′ is the straight line required.

For the triangles ADE and FDG are similar, and consequently (VI. 11. El.) AD : ED :: DF : DG; whence (V. 6. El.) ED.DF = AD.DG; but the rectangle AD.DG is equal to the given space, and therefore the rectangle ED.DF is also equal to that space.



A limitation evidently takes place, when the points F and F′ coincide, and the circle touches the straight line AB. In this case, the angle AFD or BFD, being equal to DGF in the alternate segment, is therefore equal to AED, and consequently AFE is equal to AEF, and (I. 11. El.) AF=AE.

PROP. XXVIII. PROB.

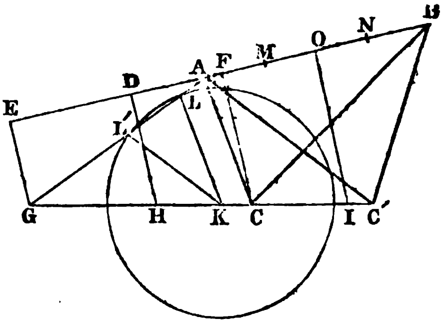
From two given points, to draw straight lines to the same point in a line given by position, which shall be together equal to a given line.

Let it be required, from the given points A and B, to draw to a point in the line HI, the straight lines AC and BC, whose sum is given.

ANALYSIS.

Join AB, and bisect it in O, make OD equal to half the sum of the converging lines AC and BC, let fall the perpendicular CF on AB, and cut off DM equal to AC.

It is obvious (II. 21. cor. El.) that the difference between the squares of AC and BC is equivalent to the difference between the squares of AF and BF, and consequently that the rectangle under the sum and difference of AC and BC is equivalent to the rectangle under the sum and difference of AF and BF; that is, the rectangle under 2OD and 2OM is equivalent to the rectangle under 2OA and 2OF. Wherefore OD.OM=OA.OF, and (V. 6. El.) OA : OD : : OM : OF; make OA : OD : : OD : OE, and (V. 20. El.) OA : OD : : OD—OM or AC : OE—OF or EF, and alternately OA : AC : : OD : EF. Draw EG, DH, and OI perpendicular to BE, and (VI. 1. El.) OD : EF : : HI : GC; consequently HI : GC : : OA : AC, or HI : OA : : GC : AC. But HI and OA being given, the ratio of GC to AC is likewise given; having joined GA, therefore, the triangle GAC has an angle GCA given, and the sides containing it in a given ratio, and is therefore (VI. 13. El.) given in species. Whence the angle ACG being given, the point C is given, and the straight lines AC and BC.



COMPOSITION.

Having joined AB, bisect it in O, and OD representing half the sum of the lines as before, make OA : OD : : OD : OE, draw EG, DH, and OI perpendicular to AB, join GA, make GK equal to HI, and from K as a centre, with a radius equal to OA, describe a circle touching or cutting GA in L, join KL, and parallel to it draw AC, and join BC; then AC and BC are the lines required.

For since (VI. 2. El.) GK or HI : KL or OA : : GC : AC or DM : by alternation OA : DM : : HI : GC, and consequently (VI. 1. El.) OA : DM : : OD : EF, or OA : OD : : DM : EF; but by construction OA : OD : : OD : OE, and therefore (V. 20. El.) OA : OD : : OD—DM or OM : OE—EF or OF; whence the rectangle OA, OF is equivalent to the rectangle OD, OM, and likewise $2OA.2OF = 2OD.2OM$. Make $ON = OM$, and $2OD.2OM = (DN + DM)(DN - DM) = DN^2 - DM^2$, and consequently $2OA.2OF =$ (II. 21. El.) $BC^2 - AC^2 = DN^2 - DM^2$; but AC is equal to DM, and therefore $BC = DN$, or $AC + BC = DM + DN = 2DO$.

PROP. XXIX. PROB.

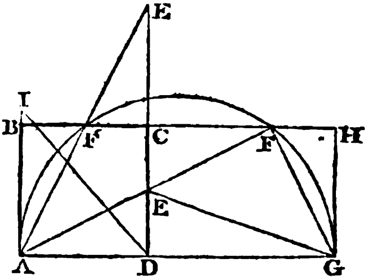
From one of the corners of a given square, to draw a straight line, such that its portion, intercepted between the opposite sides of the figure, shall be equal to a given straight line.

Let ABCD be a square, and from the point A let it be required to draw AEF, so that the part EF, intercepted between CD and BC, or their extension, may be equal to a given straight line.

ANALYSIS.

Draw FG perpendicular to AF, meeting AD produced in G, from G let fall the perpendicular GH upon BC produced, and join EG.

The angle EFH is (I. 30. El.) equal to ECF and FEC, and it is also equal to EFG and GFH; consequently, ECF and EFG being right angles, the remaining angles FEC and GFH are equal; whence the triangles EAD and FGH, having the angle AED or CEF equal to GFH, the angles at D and H both right angles, and the side AD equal to GH or CD,—are (I. 21. El.) equal, and therefore the side AE is equal to FG. But EFG and EDG being right-angled triangles, $EF^2+FG^2=EG^2=ED^2+DG^2$, (II. 10. El.), or $EF^2+AE^2=ED^2+DG^2$; but $AE^2=AD^2+ED^2$, and hence $EF^2+AD^2+ED^2=ED^2+DG^2$, or $EF^2+AD^2=DG^2$. Wherefore, since EF and AD are both given, DG is also given, and consequently AG: but the right angle AFG being contained in a semicircle described upon AG, the point F or F′, its contact or intersection with BC, is given, and consequently the straight line AEF.



COMPOSITION.

Make AI equal to the given straight line, join DI, and, equal to this, produce AD to G, upon AG describe a semicircle meeting the extension of BC in F or F′, and join AEF or AF′E′; EF, the external part of that straight line, is equal to AI.

For join FG, EG, and let fall the perpendicular GH upon BF. It is evident that $EF^2+FG^2=ED^2+DG^2$; and FG being equal to AE, $EF^2+AE^2=ED^2+DG^2$.

But $AE^2=AD^2+ED^2$, and $DG^2=DI^2=AD^2+AI^2$; whence $EF^2+AD^2+ED^2=ED^2+AD^2+AI^2$, and therefore $EF^2=AI^2$, and $EF=AI$.

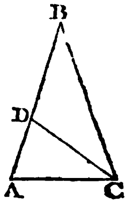
PROP. XXX. PROB.

To investigate the construction of a regular pentagon or decagon.

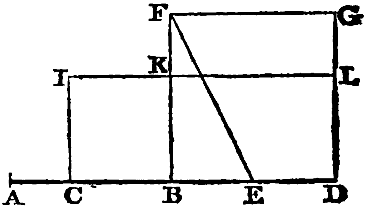
1. Every regular polygon is capable of being inscribed in a circle, and therefore the angles, formed at the centre by drawing radii to the several corners of the figure, are each of them equal to that part of four right angles corresponding to the number of sides. Consequently the central angles of a pentagon are each equal to the fifth, and those of a decagon are each equal to the tenth, part of four right angles; but an angle at the circumference being half of that at the centre, the vertical angle of the isosceles triangle, formed in the pentagon by drawing straight lines from any corner to the extremities of the opposite side, must also be the tenth part of four right angles. Whence the construction of a regular pentagon or decagon involves the description of an isosceles triangle, whose vertical angle is equal to the tenth part of four right angles, or the fifth part of two right angles.

2. Since the vertical angle of that isosceles triangle is the fifth part of two right angles. the angles at its base must be together equal to the remaining four-fifths, and each of them is consequently two-fifths of two right angles. Wherefore each of the angles at the base of that component triangle is double of its vertical angle.

3. Let ABC be such an isosceles triangle, having each of the angles at A and C double of the angle at B. Draw CD bisecting the angle ACB. The angle BCD must then be equal to CBD, and consequently the side CD is equal to BD. But in the triangles BAC and CAD, the angle ABC is equal to ACD, the angle CAB common to both, and consequently the remaining angle BCA is equal to CDA; whence CDA is equal to CAD, and therefore the side AC is equal to CD. Thus the three straight lines AC, CD, and BD are all equal. Again, because CD bisects the angle ACB, (VI. 11. El.) BC : AC :: AC : AD, that is, AB : BD :: BD : AD. Hence AB is divided in extreme and mean ratio at the point D,—or the square of BD or AC, the base of the isosceles triangle, is equal to the rectangle under the side AB and the remaining segment AD. Whence the construction of a regular pentagon or decagon depends on the medial section of a straight line.



4. Now let the straight line AB be divided by a medial section, or $BC^2 = BA.AC$. Add to each the rectangle $BA.BC$, and $BC^2 + BA.BC = BA.AC + BA.BC$, or $BC(BA+BC) = BA^2$. To AB annex BD equal to it, and $BC.CD = BD^2$. Bisect BD in E, and the straight lines CD and BC are the sum and difference of CE and BE; whence the rectangle under CD and BC, or the square of BA, is equal to the excess of the square of CE



above the square of BE, and therefore $CE^2 = BA^2 + BE^2$. Erect the perpendicular BF equal to BA, and join EF. It is evident that, $EF^2 = BA^2 + BE^2$, and consequently $EF^2 = CE^2$, and $EF = CE$; but EF being given, CE and BC are therefore given.

The composition of this general problem forms a series of the most interesting propositions in the Elements of Geometry. Art. 4. corresponds to Prop. 19. Book II.; Art. 3. to Prop. 3. and 4. Book IV.; and Art. 2. and 1. coincide with the 5th and 8th Propositions of the same Book.

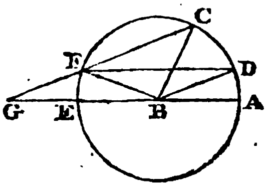
PROP. XXXI. PROB.

To discover the conditions required for the trisection of an angle.

Let ABC be an angle, of which ABD is the third part. About the vertex B describe a circle, draw DF parallel to AB, join CF, and produce it to meet the extension of A in G.

ANALYSIS.

Because the chord DF is parallel to AE, the arc EF (III. 18. El.) is equal to AD, and consequently (III. 12. cor. 1. El.) the angle EBF is equal to ABD, or is half of the remaining angle DBC; but half this angle is equal (III. 15. El.) to the angle DFC at the circumference, which (I. 22. El.) is equal to its opposite angle BGF. Wherefore the angles



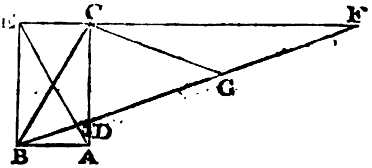
BGF and GBF are equal, and (I. 11. El.) the triangle BFG is isoscèles; and thus to solve the problem it would require, to draw CFG, such that the extreme part FG shall be equal to the radius of the circle.

Otherwise thus.

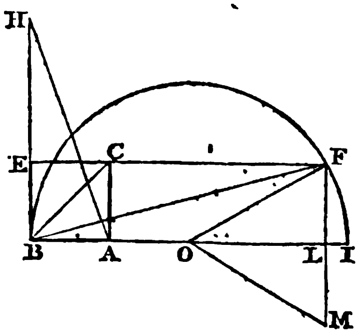
Let the angle ABD be the third part of ABC. Erect the perpendicular ADC, complete the rectangle BACE, extend the side EC to meet BD produced in F, and draw CG making the angle FCG equal to CFG.

ANALYSIS.

Because the angle FCG is equal to CFG, the side GF (I. 11. El.) is equal to GC, and the exterior angle CGB (I. 30. El.) is double of either of those angles. But the angle CBA being by hypothesis triple of ABD, the angle CBG is double of it, or of CFG, and therefore is equal to CGB; whence (I. 11. El.) the side BC is equal to GC. Again, from the right angles EBA and FCD, take away the equal angles ABD and FCG, and the remaining angles EBD and GCD are equal; but EBD is equal (I. 22. El.) to the alternate angle BDA, which is equal to the vertical angle CDF; consequently the angle GCD is equal to GDC, and therefore the side GD is equal to GC. Thus it appears, that the four straight lines BC, GC, GD, and GF, are all equal. Whence DF, the external segment of the trisecting line BF, is double of BC the diagonal of the rectangle BACE.



Scholium. Such then are the final conditions on which the trisection of an angle must depend. But to fulfil them in general, exceeds the powers of elementary geometry. In some very limited cases indeed, the trisection of an angle can be effected, merely by the help of straight lines and circles. Thus, when the proposed angle ABC is half a right angle, it may be trisected by the application of Prop. 29. of this Book. For, produce BE so that $BH = 2BC$, join AH, produce BA till $AI = AH$, and on BI describe a semicircle meeting the production of EC in F; the angle ABF is the third part of ABC.



This result agrees with what is derived from simpler views. For $BH^2 = 4BC^2 = 8BA^2$, and $AI^2 = BH^2 + BA^2 = 8BA^2 + BA^2 = 9BA^2$; whence $AI = 3BA$, the diameter $BI = 4BA$, and consequently the radius $OI = 2BA$. Let fall the perpendicular FL, and produce it equally on the other side, join OF and OM. The triangles OFL and MOL are evidently equal, and therefore OF, OM, and FM, are all equal to 2BA, or 2FL; consequently the triangle FOM is equilateral, and the angle FOM two-thirds of a right angle; the angle FOL is hence one-third of a right angle, and the angle ABF at the circumference, being the half of it, is therefore equal to the sixth part of a right angle.

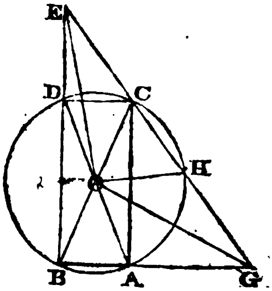
PROP. XXXII. PROB.

To investigate the conditions required in finding two mean proportionals.

Let the containing sides AB and AC of the rectangle ABCD be the extremes of a continued proportion, of which the successive mean terms are DE and AG.

ANALYSIS.

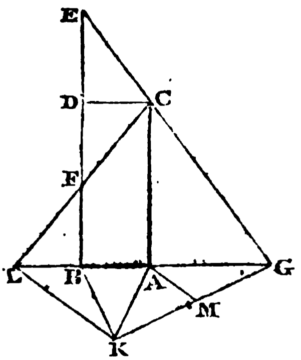
Join CE and CG. Because AB or CD : DE :: AG : AC, and CDE, being a right angle, is equal to GAC, the triangles DCE and AGC are (VI. 13. El.) similar; whence the angle DEC is equal to ACG, and the angles ACG and ACE equal to DEC and ACE, or (I. 22. El.) two right angles, and consequently ECG forms a straight line. Draw the diagonals BC, AD, and join their intersection O with the points E and G. The triangles BOD and BOA being evidently isosceles, therefore (II. 20. El.) $OE^2 = OD^2 + BE.ED$ and $OG^2 = OA^2 + BG.GA$; but (VI. 11. El.) BG : BE :: GA : AC, or DE : GA, and hence (V. 6. El.) $BE.DE = BG.GA$. Wherefore, OD being equal to OA, the square of OE is equal to that of OG, and consequently the point O is equidistant from E and G. Hence, likewise, if a circle were described about the given rectangle, the intercepted segment EC (IV. 4. cor. El.) would be equal to GH.



The solution of the problem, then, requires to draw ECG, such that the distance OE be equal to OG, or that the part EC without the circle be equal to the opposite part GH.

Otherwise thus.

The first part of the construction remaining the same, it was proved that the rectangle BE.ED is equivalent to BG.GA; bisect BD in F, and BE.ED $+$ DF^2, or (II. 17. cor. 2. El.) EF$^2 =$ BG.GA $+$ DF^2. On AB construct the isosceles triangle BKA, having each of its sides BK and AK equal to DF, and join GK; then (II. 20. El.) BG.GA $+$ AK$^2 =$ GK^2,

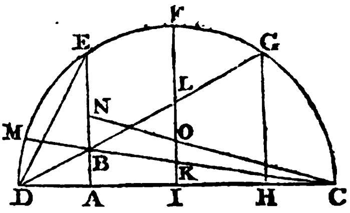


and consequently EF$^2 =$ GK^2, or EF $=$ GK. But, by hypothesis, AB : DE : : DE : : GA : : GA : AC, and (V. 11. El.) DC or AB : GA : : DE : AC, or (V. 13. El.) 2AB : GA : : 2DE : AC; join CF and produce it to meet the extension of AB in L; the triangles CFD and LFB (I. 20. El.) are evidently equal, and CD or AB equal to BL. Wherefore AL is to GA as 2DE to AC or BD, or (V. 3. El.) as DE to DF the half of BD, and consequently (V. 9. El.) GL : GA : : EF : DF. Join LK and draw AM parallel to it; then (VI. 1. El.) GL : GA : : GK : GM, whence EF : DF : : GK : GM; but EF $=$ GK, and therefore DF $=$ GM. Now the points F, L and K are evidently given, and consequently the straight line LK and its parallel AM are given in position.

To effect, therefore, the construction of the problem, it is required from the point K to draw the straight line KMG, such that the part MG, intercepted between AM and BA produced, shall be equal to the half of AC.

Or thus.

Let AB and AC, the extreme terms of the continued proportion, stand as before at right angles, and having produced CA to D, let AB : AD : : AD : AE : : AE : AC. Since, then, AD : AE : : AE : AC, it follows (V. 6. El.) that $AD.AC = AE^2$; whence (III. 26. cor. 1. El.) the point E lies in the circumference of a semicircle described upon CD. Join DE, produce DB to the circumference, and draw the perpendicular radius IF. Because AB : AD : : AD : AE, and the angle DAE is common to the two triangles BAD and DAE—these triangles (VI. 13. El.) are similar; consequently the angle ADB is equal to AED, and (III. 18. cor. El.) the arc CG is equal to DE; whence the arc FG is equal to FE, and (III. 10. El.) the segment IH of the diameter equal to IA, or the oblique line GL (VI. 1. El.) is equal to LB.



On this condition, therefore, that GD shall have its intercepted portion GL equal to LB, or that the perpendiculars EA and GH shall be equidistant from the centre, the solution of the problem depends. The ratio of KI to IC is evidently the same as that of AB to AC. Wherefore a semicircle being described with the radius IC—could a straight line BD be drawn from D, such that the part BG, intercepted between the circumference and the straight line CKM drawn from the other extremity of the diameter, be bisected in L by the perpendicular radius IF—the problem would be solved: For make AN=AD, and join CN meeting IF in O; it is manifest, from what has been shown, that IK, IO, IL, and IC are continued proportionals.

GEOMETRICAL ANALYSIS.

BOOK II.

DEFINITION.

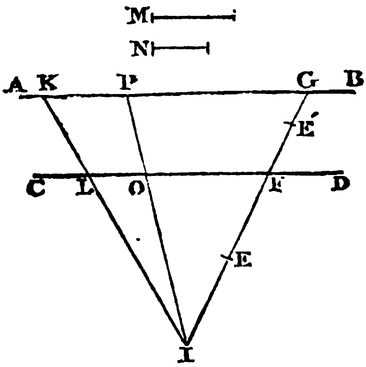
A VARIABLE quantity derived from another given or constant quantity, or which depends on it by some relation according to a given law, is necessarily confined between certain extreme limits. When it has acquired the greatest possible expansion, it is said to have reached a *maximum*; and when it has contracted into its lowest dimensions, it occupies the state of a *minimum*.

PROP. I. PROB.

From a given point, to draw a straight line intercepting, on two given parallels, segments which shall have a given ratio.

Let AB and CD be two parallels, in which are two given points, P and O; and let it be required, from another given point E, to draw EF, such that PG shall be to OF in the ratio of M to N.

ANALYSIS.



Join PO, and produce it to meet EF, or its extension in I.

Because PG and OF are parallel, PI : OI :: PG : OF (VI. 2. El.); but the ratio of PG to OF is given, and hence that of PI to OI, and of PO to OI, are given. And since PO is given, OI and the point I are given; wherefore IEF, and the segments PG and OF are given.

COMPOSITION.

Make PK = M and OL = N, join KL, PO, and produce them to meet in I, and draw IEF; PG and OF are the required segments.

For (VI. 2. El.) the parallels AB and CD being cut proportionally by the diverging lines IK, IP, and IG,—PG is to OF as KP to OL, that is, as M to N.

If M be equal to N, the point I vanishes, and EF becomes evidently a parallel to OP.

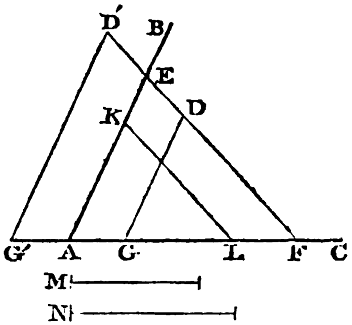
If the straight lines KL and PO meet in the given point E, the problem is by its nature indeterminate, or it admits of indefinite solution; for, in that case, the segments PG and OF, intercepted by any straight line whatever, drawn through E, have all the same ratio.

PROP. II. PROB.

Two diverging lines being given in position, to draw, through a given point, a straight line, intercepting segments which shall have a given ratio.

Let it be required, through D, to draw EDF, so that AE shall be to AF in the ratio of M to N.

ANALYSIS.



Through D, (I. 23. El.) draw DG parallel to AE, and meeting AC or its production, in G.

The triangles EAF and DGF are similar, and therefore (VI. 11.) AE : AF : : GD : GF; but the ratio of AE to AF is given, and consequently that of GD to GF. And since GD and the point G are evidently given, GF and the point F are likewise given.

COMPOSITION.

From AB and AC cut off $AK=M$, and $AL=N$, join KL, and parallel to it draw EDF through D; AE and AF are the segments required.

For (VI. 1. El.) the parallels EF and KL cut the diverging lines AB and AC proportionally, and therefore AE is to AF, as AK to AL, that is, as M to N.

PROP. III. PROB.

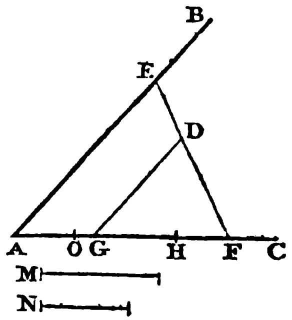
Two diverging lines being given in position, to draw through a given point, a straight line cutting off segments—on the one from their intersection, and on the other from a given point,—that shall have a given ratio.

Let AB and AC be two diverging lines, it is required, through the point D, to draw EDF, so that AE shall be to the part OF, in the ratio of M to N.

ANALYSIS.

Draw DG parallel to AE, and meeting AC, or its production in G, and make AE : GD : : OF : OH.

By alternation, AE : OF : : GD : OH; but the ratio of AE to OF is given, and thence that of GD to OH; and since GD and the point O are given, OH and the point H are also given. Again, because AE : GD : : OF : OH, and (VI. 2. El.) AE : GD : : AF : GF, it follows that OF : OH : : AF : GF: whence (V. 10. El.) FH : OH : : AG : GF, and (V. 6. El.) GF.FH = AG.OH. But AG and OH are both given, and consequently the rectangle under the segments GF and FH of the given portion GH is also given, and thence the point of section F is given, and the straight line ED.



COMPOSITION.

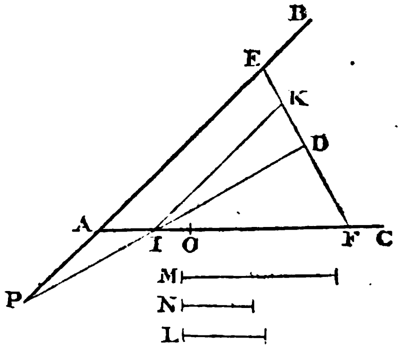
Make GD to OH, as M to N, and (VI. 19.) divide GH in F, so that the rectangle GF, FH shall be equal to AG OH, and draw EDF; then the segment AE is to OF as M to N. Since GF.FH = AG.OH, therefore FH : OH : : AG : GF, and (V. 9. El.) OF : OH : : AF : GF; but (VI. 2. El.) AE : GD : : AF : GF, and consequently AE : GD : : OF : OH, and alternately AE : OF : : GD : OH, that is, in the given ratio.

PROP. IV. PROB.

Two diverging lines being given in position, to draw, through a given point, a straight line, cutting off segments from given points in a given ratio.

Let AB and AC be two diverging lines; it is required, through the point D, to draw EDF, so that PE shall be to OF in the ratio of M to N.

ANALYSIS.



Join DP cutting AF in I, and, through I, draw IK parallel to AB, and meeting EF in K.

Because the points D and P are given, the straight line DP is given in position, and consequently its intersection I with AC is given, whence IK, being parallel to AB, is likewise given in position. But (VI. 2. El.)

PE : IK : : PD : ID, and since PD and ID are both given, the ratio of PE to IK is given; consequently, the ratio of PE to OF being given, the ratio of IK to OF is given. Wherefore, by the last proposition, the straight line KDF is given in position.

COMPOSITION.

Join PD and draw IK parallel to AB, make M to L, as PD to ID, and draw, by the last proposition, KDF, so that IK shall be to OF, as L to N; then will PE and OF be the segments required.

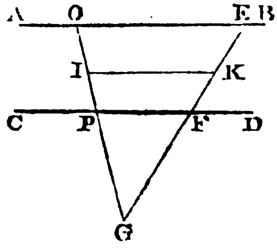
For (VI. 2. El.) PE : IK : : PD : ID : : M : L, and IK : OF : : L : N; whence (V. 16. El.) PE : OF : : M : N.

PROP. V. PROB.

Two parallels being given, from a point in a given intersecting line, to draw another straight line cutting off segments which shall contain a given rectangle.

Let AB, CD be two parallels, and G a given point, through which it is required to draw FE intercepting, from given points O and P in the same direction OPG, segments OE and PF, that shall contain a given rectangle.

ANALYSIS.



Because AB and CD are parallel, GO : GP : : OE : PF (VI. 2. El.) and consequently (V. 25. cor. 2. El.) GO : GP : : OE^2 : OE.PF; and GO and GP being given, their ratio is given, and therefore the ratio of OE^2 to OE.PF is given; but the rectangle OE, PF is given, and hence the square of OE and consequently OE itself, are given.

COMPOSITION.

Find (VI. 16. El.) GI, a mean proportional between GO and GP, draw IK parallel to AB or CD, and such (III. 27. El.) that its square shall be equal to the given rectangle, and join EKFG; this is the straight line required.

For OE, IK, and PF being parallel, OG : IG : : OE : IK, and PG : IG : : PF : IK (VI. 2. El.); whence compounding these analogies (V. 22. El.) OG.PG : IG^2 : : OE.PF : IK^2; but OG.PG = IG^2, and consequently (V. 4.) OE.PF = IK^2.

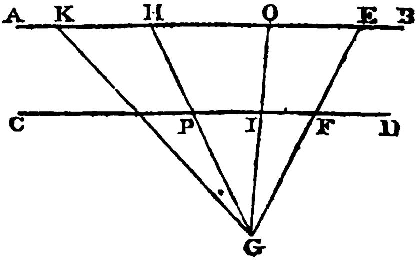
PROP. VI. PROB.

Through a given point, to draw a straight line intercepting, from given points on two given parallels, segments which shall contain a given rectangle.

Let AB and CD be parallels in which the points O and P are given, and let it be required through G to draw GFE, so that the segments OE and PF shall contain a given rectangle.

ANALYSIS.

Draw GO and GP, cutting the parallels in I and H. Because the points O, P, and G are given, the straight lines GIO and GPH are given in position, and consequently their intersections I and H with the given paral-



lels. And since AB is parallel to CD, GP:GH::PF:HE (VI. 2. El.) but (V. 25. cor. 2. El.) PF:HE::PF.OE:HE.OE, and consequently GP:GH::PF.OE:HE.OE. Now, GP and GH being given, their ratio is given, and hence that of PF.OE to HE.OE; wherefore the rectangle PF, OE being given, the rectangle under the segments HE and OE of the given straight line HO is likewise given; whence (VI. 17. El.) the point E is given, and consequently the straight line GFE.

COMPOSITION.

Draw GO and GP, find (II. 9. El.) HK the side of a rectangle GP, HK which is equal to the given space, and (VI. 17. El.) divide HO in the point E, so that the rectangle under its segments HE and OE shall be equal to the rectangle HG, HK, and join GFE; this is the straight line required.

For HE : PF : : HG : GP, and hence (V. 25. cor. 2. El.) HE.OE : PF.OE : : HG.HK : GP.HK; but, by construction, the rectangle HE.OE is equal to GH.HK, and consequently (V. 8. and 4. El.) PF.OE=GP.HK, or the given space.

PROP. VII. PROB.

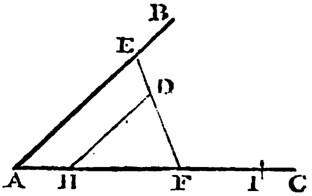
To draw through a given point a straight line, cutting from two given diverging lines, segments which shall contain a given rectangle.

Let AB and AC be two diverging lines given in position, and let it be required through the point D, to draw EDF, so that the rectangle under the segments AE, AF shall be equal to a given space.

ANALYSIS.

Draw HD parallel to AB, and make (II. 9. El.) the rectangle DH.AI equal to the given space.

Because AE.AF=DH.AI, AE : DH : : AI : AF (V. 6. El.), but AE : DH : : AF : FH (VI. 2. El.), and therefore AF : FH : : AI : AF; whence (V 11. cor. El.) AH : AF : : IF : AI, and (V. 6. El.) AH.AI = AF.IF. Now DH, being parallel to AB, is given, and consequently AI is given; wherefore the rectangle AH, AI being given, AF.IF is also given; and since AI is given, its internal or external section is (VI. 17. El.) given.



COMPOSITION.

Draw DH parallel to AB, find (II. 9. El.) AI, which contains with DH a rectangle equal to the given space, and divide AI (VI. 17. El.) so that the rectangle under its segments AF, FI shall be equal to the rectangle AI, AH; EDF is the straight line required. For, by construction, AF.IF=AI.AH, whence (V. 6. El.) AH : AF : : IF : AI, and (V. 10 and 7. El.) AF : FH : : AI : AF; but AF : FH : : AE : DH, and consequently AE : DH : : AI : AF, and (V. 6. El.) AE.AF=DH.AI.

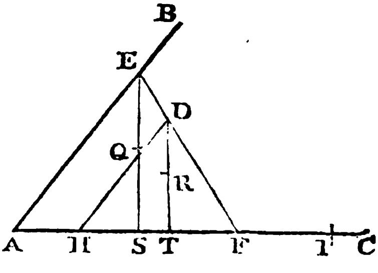
PROP. VIII. PROB.

Through a given point to draw a straight line, which shall, by its intersection with two given diverging lines, form a triangle containing a given space.

Let it be required, through the point D, to draw a straight line, EDF intercepting, between the diverging lines AB and AC, a triangle AEF, which shall contain a given space.

ANALYSIS.

Draw DH parallel to AB, upon AC let fall the perpendiculars ES and DT, and find (II. 9. and 7. El.) under the altitude DT, AI the base of a triangle, containing the given space.



Because the rectangles ES, AF and DT, AI are (II. 5. El.) each double of the triangles AEF and ADI, they are equal, and consequently (V. 6. El.) ES : DT : : AI : AF. But the triangles AES and HDT are evidently similar, and therefore (VI. 11 El.) AE : ES : : HD: DT, or alternately AE : HD : : ES : DT; whence AE : HD : : AI : AF, and AE.AF = HD.AI. Now HD is given, and consequently AI; wherefore the rectangle AE, AF is given and thence, by the last proposition, the straight line EDF is given in position.

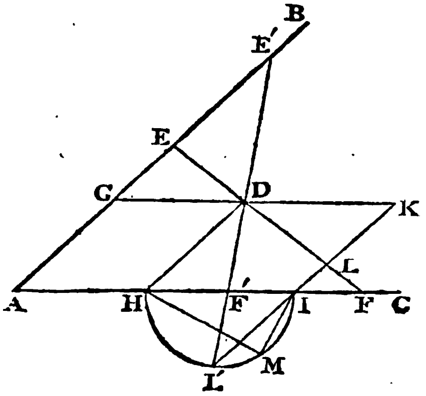
COMPOSITION.

Draw DH parallel to AB, let fall the perpendicular DT, bisect this in the point R, find (II. 9. El.) the side AI, which with RT contains a rectangle equal to the given space, and, by the last proposition, draw EDF, such that the rectangle AE.AF shall be equal to DH.AI.

Having let fall the perpendicular ES, and bisected it in Q, the triangles AES and HDT are similar; whence AE : ES : : HD : DT, and alternately AE : HD : : ES : DT, or (V. 3. El.) AE : HD : : QS : RT; wherefore AE.AF : HD.AI : : QS.AF : RT.AI; but the rectangle AE.AF = HD.AI, and hence (V. 4. El.) QS.AF = RT.AI, or the triangle AEF is equal to the given space.

This problem will admit of a simpler construction, in the case where the given point D lies between the diverging lines AB and AC. For draw DG parallel to AC, and make (II. 9. El.) the rhomboid AGKI equal to the given space.

Because the triangle AEF is equal to the rhomboid AGKI, take each of them from the compound figure AEDKLF, and the triangles GED and ILF will remain equivalent to the triangle DLK; but these supplementary triangles, being formed by parallel lines, are evidently similar, and consequently the homologous sides GD and IF are (VI. 26. El.) sides of a right-



angled triangle, of which DK is the hypotenuse; wherefore (II. 10. El.) $GD^2+IF^2=DK^2$, or GD and DK being equal to AH and HI, (I. 26. El.) $IF^2=HI^2-AH^2$. And since HI and AH are both given, it follows that IF is given.

COMPOSITION.

Construct (II. 8. El.) the rhomboid AGKI equal to the given space, draw DH parallel to AB, on HI describe a semicircle, in which inflect HM equal to AH, join IM, and make IF, or IF′, equal to it; EDF, or E′DF′, is the base of the required triangle.

For (III. 19. El.) HMI being a right angle, $IH^2=HM^2+IM^2$ (II. 10. El.), or $DK^4=GD^2+IF^2$; whence (VI. 26. El.) the triangle DLK, or DL′K, is equivalent to the triangles GED and ILF, or to GE′D and IL′F; and, adding to both the excess of the rhomboid AK above the triangle DLK, or DL′K, the rhomboid AK is equivalent to the triangle AEF or AE′F′, which is, therefore, equal to the given space.

PROP. IX. PROB.

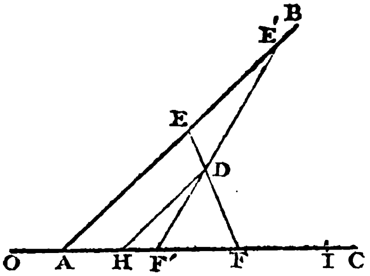
Through a given point to draw a straight line, cutting off segments, from two given diverging lines—on the one from their intersection, and on the other from a given point—which shall contain a given rectangle.

Let it be required to draw EDF, so that the rectangle AE, OF shall be equal to a given space.

ANALYSIS.

Draw DH parallel to AB, and (II. 9. El.) make the rectangle DH.OI equal to the given space; OI and the point

I are, therefore, given. And since AE.OF = DH.OI, it follows that AE : DH : : OI : OF; but (VI. 2. El.) AE : DH : : AF : FH, and consequently AF : FH : : OI : OF. Wherefore (V. 11. El.) AF : AH : : OI : FI, and (V. 6. El.) AF.FI = AH.OI; hence AI and the rectangle under its segments, AF and FI, are given, and consequently (VI. 17. El.) the point of section F and the straight line EDF are given.



COMPOSITION.

Having drawn DH parallel to AB, and made the rectangle DH.OI equal to the given space, divide AI (VI. 17. El.) in F, or F', such that the rectangle under its segments shall also be equal to the rectangle AH.OI; EDF, or E'DF', is the required straight line. For since AF.FI = AH.OI, AF AH : : OI : IF; whence (V. 11. El.) AF : FH : : OI : OF; but (VI. 2. El.) AF : FH : : AE : DH, and, therefore, AE : DH : : OI : OF, and the rectangle AE.OF is equal to DH.OI, or the given space.

PROP. X. PROB.

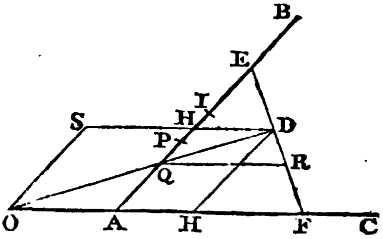
Through a given point, to draw a straight line, cutting off segments from given points, on two given diverging lines, that shall contain a given rectangle.

Let it be required to draw EDF, so that the rectangle OF.PE shall be equal to a given space.

ANALYSIS.

Join DO meeting AE in Q, and draw QR parallel to AC.

Because (VI. 2. El.) DO : DQ : : OF : QR, it follows (V. 25. cor. 2. El.) DO : DQ : : OF.PE : QR.PE; but DO and DQ are evidently given, and therefore the rectangle OF, PE has to QR, PE a given ratio; and since OF.PE is given, the rectangle QR, PE is likewise given, and QR, being parallel to AC, is given in position. Whence, by the last proposition, the intersecting line EDR or EDF, is given in position.



COMPOSITION.

Join DQO, draw DH parallel to AC, and produce it meeting in S the parallel to AB, make the rectangle DS.PI equal to the given space, and divide PI in E, such that the rectangle under its segments PE, IE shall be equal to the rectangle AH, PI; EDF is the straight line required.

For DQ : DO : : DH : DS : : QR : OF, and consequently (V. 25. cor. 2. El.) DH.PI : DS.PI : : PE.QR : PE.OF; but, by the last proposition, DH.PI = PE.QR, whence the rectangle DS, PI, or the given space, is equal to the rectangle PE, OF.

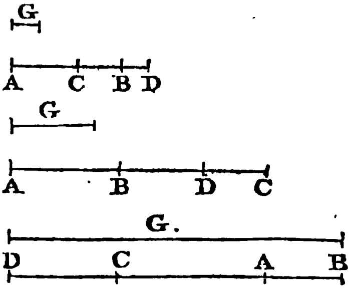
PROP. XI. PROB.

To divide a given straight line, so that the rectangle under one of its segments and a given line, shall be equal to the square of the other segment.

Let it be required to divide AB in C, such that the rectangle under AC and G shall be equal to the square of CB.

ANALYSIS.

Make BD = G, and since $AC.G = CB^2$, it follows (V. 6. El.) that AC : CB :: CB : BD, and consequently (V. 9. and 10. El.) AB : CB :: CD : BD; whence (V. 6. El.) AB.BD = CB.CD. But the rectangle AB, BD is given, and, therefore, the rectangle CB, CD is also given; and BD being given, the point of section C is (VI. 17. El.) thence given.



COMPOSITION.

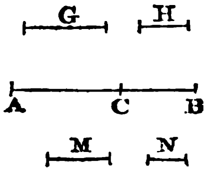
In the same straight line AB, make BD equal to G, and (VI. 17. El.) cut BD such that the rectangle CB.CD be equal to AB.BD; C is the point of section required. For it is evident (V. 6. El.) that AB : CB :: CD : BD, and consequently (V. 10. El.) AC : CB :: CB : BD; wherefore (V. 6. El.) AC.BD, or $AC.G = CB^2$.

PROP. XII. PROB.

To divide a given straight line, so that the rectangle under one of its segments and a given line shall have a given ratio to the square of the other segment.

Let it be required to divide AB in C, such that $AC.G : CB^2 :: M : N$.

ANALYSIS.



Make (VI. 3. El.) $G : H :: M : N$, and H is given; but $AC.G : CB^2 :: G : H$, and consequently (V. 25. cor. 2. El.) $CB^2 = AC.H$; wherefore, by the last proposition, the section of AB is given.

COMPOSITION.

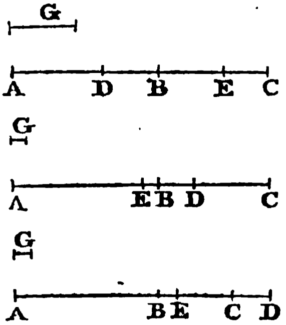
Having made $M : N :: G : H$, let AB be divided by the last proposition, so that $AC.H = CB^2$; then $AC.G : CB^2 :: M : N$. For $AC.G : AC.H$ or $CB^2 :: G : H$, or $M : N$.

PROP. XIII. PROB.

In the same straight line, three points being given, to find a fourth point, such that the rectangle under its distance from the first and a given line, shall be equal to the rectangle under its distances from the second and third points.

Let it be required to find the point D, so that AD.G =CD.BD.

ANALYSIS.



Make BE=G, and because AD.G = CD.BD, it follows that AD : CD : : BD : BE; whence (V. 9. and 10. El.) AC : CD : : DE : BE, and AC.BE = CD.DE. But the rectangle AC, BE being evidently given, the rectangle under the segments CD, DE of CE, a given straight line, is also given, and consequently (VI. 17. El.) the point of section D is given.

COMPOSITION.

Having made BE=G, divide (VI. 17. El.) CE in D, so that CD.DE=AC.BE; D is the point required.

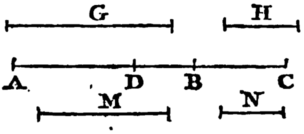
For (V. 6. El.) AC : CD :: DE : BE, and (V. 10. El.) AD : CD : : BD : BE; whence AD.BE, or AD.G = CD.BD.

PROP. XIV. PROB.

In the same straight line, three points being given, to find a fourth, so that the rectangle under its distance from the first and a given line, shall have a given ratio to the rectangle under its distances from the second and third points.

Let it be required to find a point D, such that AD.G : CD.BD : : M : N.

ANALYSIS.



Make M : N : : G : H, whence H is given; but since AD.G : CD.BD : : G : H, it is evident that AD.H = CD.BD; wherefore, by the last proposition, the point of section D is given.

COMPOSITION.

Having made G : H : : M : N, find, by the last proposition, the point D, so that CD.BD = AD.H; D is the section required. For (V. 25. cor. 2. El.) AD.G : AD.H or CD.BD : : G : H, or M : N.

PROP. XV. PROB.

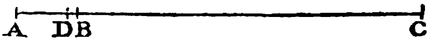
In the same straight line, three points being given, to find a fourth, so that the square of its distance from the first shall be equal to the rectangle under its distances from the second and third points.

Let it be required to find a point D, such that AD^2 = CB.BD.

1. *When the point D lies between A and B.*

ANALYSIS.

Because AD^2 = CD.BD, it follows (V. 6. El.) that CD : AD : : AD : BD; whence (V. 9. El.) AC : AD : : AB : BD, and alternately AC : AB : : AD : BD. But the ratio of AC : AB being given, the



ratio of AD to BD is given; and since AB is given, the point D (VI. 4. El.) is given.

COMPOSITION.

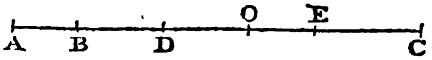
Divide AB (VI. 4. El.) in the ratio of AC to AB, and the point of section D is that required.

For, because AD : BD : : AC : AB, AD : BD : : AC—AD, or CD : AB—BD or AD (V. 19. cor. 1. El.); whence (V. 6. El.) $AD^2 = BD.CD$.

2. *When the point D lies between B and C.*

ANALYSIS.

Make DE = AD, and since $AD^2 = CD.BD$, CD : AD or DE : : AD : BD, and therefore (V. 10. El.) CE : DE : : AB : BD, and alternately CE : AB : : DE : BD, or (V. 3. El.) CE : AB : : 2DE or AE : 2BD; whence (V. 19. El.) CE : AB : : CE + AE or AC : AB + 2BD, or BE, and consequently (V. 6. El.) CE.BE = AB.AC; but the rectangle AB, AC being given, the rectangle CE, BE is likewise given, and BC being given, the point E is given (VI. 17. El.), and therefore D, the bisection of AE, is given.



COMPOSITION.

Divide BC (VI. 19. El.) in E, such that CE.BE = AB.AC, and bisect AE in D; then $AD^2 = CD.BD$.

For since CE.BE = AB.AC, it is evident that CE : AB : : AC : BE; whence (V. 17. El.) CE : AB : : AE : 2BD or (V. 3. El.) DE : BD; and alternately, CE : DE : : AB : BD, and (V. 9. El.) CD : DE or AD : : AD : BD; wherefore (V. 6. El.) $CD.BD = AD^2$.

This last case is evidently subject to limitation; for the rectangle AB, AC being equal by construction to CE.BE, must not exceed the square of the half of BC, which (II. 17. cor. 1. El.) is the greatest rectangle contained under the segments of BC. Consequently, if E coincide with the middle point O, it limits the problem; but then AB.AC $=BO^2$, or AB.AC $+ BO^2 =$ (II. 17. cor. 2. El.) $AO^2 = 2BO^2$, and therefore AO is the diagonal of a square described on BO. Whence AB : BC : : $\surd 2 - 1 : 2$, or $1 : 2 + \surd 8$; that is, the ratio of AB to BC has attained its *maximum*, when it is that of half the side of a square to the sum of the side and the diagonal.

PROP. XVI. PROB.

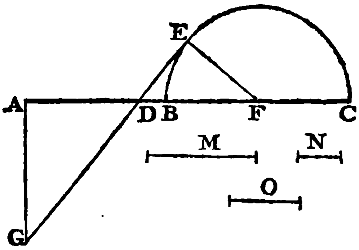
In the same straight line, three points being given, to find a fourth, such that the square of its distance from the first, shall have a given ratio to the rectangle under its distances from the second and third points.

Let it be required to find a point D, such that AD^2 shall be to CD.DB in a given ratio.

1. *When D lies between A and B.*

ANALYSIS.

On BC describe a semicircle, and draw the tangent DE; then (III. 26. cor. 2. El.) $DE^2 =$ CD.DB, and consequently the square of AD is to the square of DE in the given ratio; whence the ratio of AD to DE is given. Draw the radius EF, and produce ED to meet AG a perpendicular to AC. The triangles ADG and EDF are evidently similar, and therefore AD : AG : : DE : EF, or alternately AD : DE : : AG : EF; and since the ratio AD to DE is given, the ratio of AG to EF is also given, and the radius EF being given, AG and the point G are thence given; wherefore the tangent GE and its intersection D with AC are given.



COMPOSITION.

Let M : N be the given ratio, and to these find (VI. 16. El.) a mean proportional O, on BC describe a semicircle, make O : M : : BF : AG, a perpendicular erected from A, and (III. 22. El.) draw the tangent GDE; the intersection D is the point required.

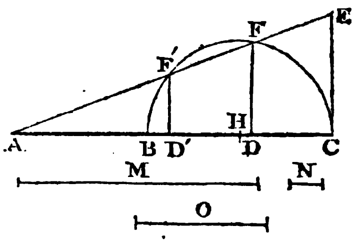
For, the triangles DAG and DEF being similar, AD : AG : : DE : EF, and alternately AD : DE : : AG : EF, or M : O; wherefore (V. 22. cor. 1. El.) $AD^2 : DE^2$: : $M^2 : O^2$, that is, (V. 24. El.) M : N; but (III. 26. cor. 2. El.) $DE^2 = CD.DB$, and consequently AD^2 : CD.DB : : : : M : N.

2. *When D lies between the points B and C.*

ANALYSIS.

On BC describe a semicircle, draw DF perpendicular to the diameter, and meeting the circumference in F, and join AF.

Because (III. 26. cor. 2. El.) $BD.DC = DF^2$, the ratio of AD^2 to DF^2 is given, and consequently that of AD to



DF; but the angle ADF, contained by these sides, being a right angle, is given, and therefore the triangle AFD is given in species. Hence the angle DAF is given, and the straight line AF given in position; wherefore the intersection F or F′, the perpendicular FD, or F′D′, and the point D, or D′, are all given.

COMPOSITION.

Let M : N express the given ratio, and to these find (VI. 16. El.) a mean proportional O, make (VI. 3. El.) M to O as AC to the perpendicular CE, join AE meeting the circumference of a semicircle described on BC in the point F or F′, and let fall the perpendicular FD or F′D′; then M : N : : AD^2 : BD.DC or AD'^2 : BD.D′C.

For the triangle ACE is evidently similar to ADF or AD′F′, and therefore AC : CE : : AD : DF, and $AC^2 : CE^2$: : $AD^2 : DF^2$: but (V. 24. El.) M : N : : $M^2 : O^2$, or as $AC^2 : CE^2$, and consequently $AD^2 : DF^2$, that is BD.DC, : : M : N.

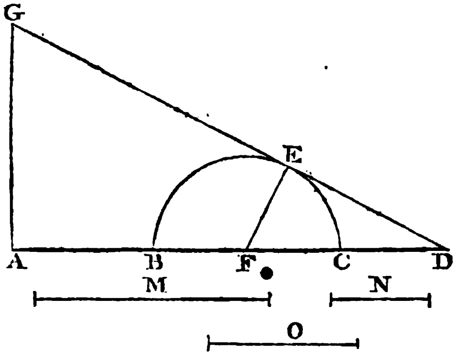
This problem evidently requires limitation; for if AE should diverge too much from AC, it will not meet the circumference at all. Hence an extreme case will occur, when AE touches the circle. But the ratio of AC to CE, or of AD to DF, will then be the same as that of a tangent from A is to the radius HB; and consequently the limiting ratio is the duplicate of this,—or the ratio of M to N can never approach nearer to the ratio of equality than that of AB.AC, or $AH^2 - HB^2$, to HB^2.

3. *When the point D lies beyond B and C.*

ANALYSIS.

On BC describe a semicircle, draw the tangent DE, and produce it to meet the perpendicular AG, and join E with the centre F.

Because (III. 26. cor. 2. El.) $BD.DC = DE^2$, the ratio of AD^2 to DE^2 is given, and consequently that of AD to DE. But the angle DEF, being (III. 24. El.) a right angle, is equal to DAG, and the angle at D is common to the triangles DGA and DFE, which are therefore similar, and hence AD : AG : : DE : EF, or alternately AD : DE : : AG : EF. And since the ratio of AD to DE is given, that of AG to EF is also given, and EF, the half of EC being given, AG and the point G are thence given. Wherefore the tangent GE and its intersection D with AC are given.



COMPOSITION.

Let M : N be the given ratio, and find the mean proportional O; make O : M : : BF : AG, a perpendicular to AC, and draw (III. 22. El.) the tangent GED; then M : N : : AD^2 : BD.DC.

For join EF. Because the triangles ADG and EDF are similar, AG : AD : : EF : ED, and alternately AG : EF : : AD : ED; but AG : EF : : M : O, and therefore M : O : : AD : ED, and $M^2 : O^2$: : $AD^2 : ED^2$, that is, M : N : : $AD^2 : ED^2$ or BD.DC.

PROP. XVII. PROB.

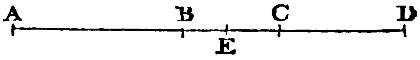
In the same straight line, four points being given, to find a fifth, such that the rectangle under its distances from the first and second points, shall have a given ratio to the rectangle under its distances from the third and fourth.

Let it be required to find a point E, so that AE.EB : DE.EC : : M : N.

1. *Let M : N be a ratio of equality.*

ANALYSIS.

Because AE.EB = DE.EC, it is manifest that AE : CE : : DE : EB; whence (V. 9. and 8. El.) AC : BD : : CE : EB, and (V. 9. El.) AC + BD : BD : : BC : EB; but the ratio of AC + BD to BD is given, whence that of BC to EB, and therefore BE and the point E, are given.



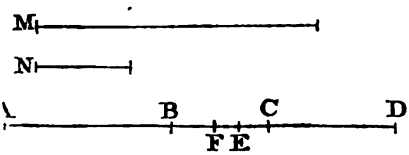
COMPOSITION.

Make AC + BD : BD : : BC : EB, and E is the point required. For (V. 10. El.) AC : BD : : CE : EB, and (V. 19. cor. 1. El.) AE : ED : : CE : EB, and hence (V. 6. El.) AE.EB = CE.ED.

2. *Let $M : N$ be a ratio of majority or minority.*

ANALYSIS.

Find, by the preceding construction, a point F, such that $AF.FB = DF.FC$.



Because $AE.EB : DE.EC :: M : N$, it follows that $AE.EB : AE.EB - DE.EC :: M : M-N$; but $AE.EB - DE.EC = (AE.EB - AF.FB) + (DF.FC - DE.EC)$, that is, $= EF(AF + BE) + EF(DF + CE)$, or $= EF(AD + BC.)$ Wherefore $AE.EB : EF(AD+BC) :: M : M-N$; consequently the point E is assigned by Prop. 14. of this Book.

The composition of the problem is thence easily derived, by retracing the steps.

PROP. XVIII. PROB.

In the same straight line, four points being given to find a fifth, such that the rectangle under its distances from the extreme points shall have a given ratio to the rectangle under its distances from the mean points.

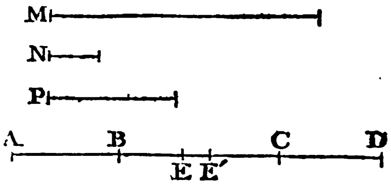
Let it be required to find a point E, so that $AE.ED : BE.EC :: M : N$.

1. *Let $AB = CD$.*

ANALYSIS.

Because $AE.ED = (AB + BE)(AB + EC)$, it is evident that $AE.ED = AB.AC + BE.EC$, whence $AE.ED$;

$AB.AC :: M : M-N$. The ratio of $AE.ED$ to $AB.AC$ is therefore given, and the rectangle under AE and ED, the segments of AD, being thus given, the point E is assigned by VI. 17. of the Elements.



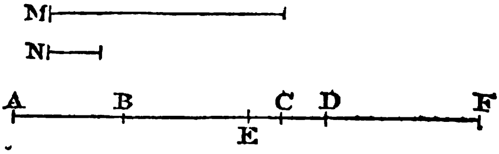
COMPOSITION.

Make $M-N : M :: AB : P$, and (VI. 19. El.) cut AD in E or E′, such that $AE.ED = P.AC$; E is the point required. For (V. 7. El.) $M : M-N :: P : AB$, and hence (V. 25. cor. 2. El.) $M : M-N :: P.AC$, or $AE.ED : AB.AC$; consequently $M : N :: AE.ED : AE.ED - BA.AC$, or $BE.EC$.

2. *Let AB and CD be unequal.*

ANALYSIS.

Because $AE.ED = (BE+AB)(EC+CD) = BE.EC + BE.CD + AB.ED$, consequently $AE.ED - BE.EC = BE.CD + AB.ED = BD.CD - ED.CD + AB.ED = BD.CD + (AB-CD)\,ED$. Produce AD to F, so that $(AB-CD)\,DF = BD.CD$; and since AB, CD and BD are all given, DF and the point F are given. Thus from construction $AE.ED - BE.EC = (AB-CD)(DF+ED) = (AB-CD)\,EF$. Now, since the ratio of $AE.ED$ to $BE.EC$ is given, the ratio of $AE.ED$ to $EF\,(AB-CD)$ is also given; wherefore $AB-CD$ being given, and the points A, C, and F, the point E is given by Prop. 14.



Applying that proposition, the construction of the problem is easily obtained.

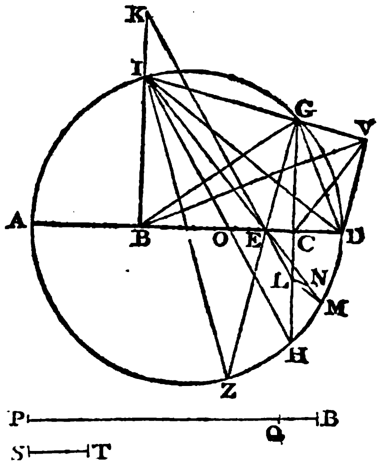
It yet remains to assign the limitations of this problem.

On AD describe a circle, erect the perpendiculars BI and GCH, join IOH, and, parallel to this, draw KEL through the point of section E, join OG, EG, and IE, which produce to the circumference, and join MG and ML.

The point O is evidently given. But the ratio of AE.ED to BE.EC may be considered as compounded of the ratio of AE.ED, or (III. 26. El.) IE.EM, to KE.EL, and of the ratio of KE.EL to BE.EC.

Now, since BK and CL are parallel, KE : EL :: BE : EC, or alternately KE : BE :: EL : EC, and therefore (V. 22. El.) $KE^2 : BE^2 :: KE.EL : BE.EC$. Again, KE and IO being parallel, KE : IO :: BE : BO, or alternately KE : BE :: IO : BO, and hence $KE^2 : BE^2 :: IO^2 : BO^2$. Wherefore $IO^2 : BO^2 :: KE.EL : BE.EC$, and consequently the ratio of these rectangles is given; let it be that of PQ to ST.

The angle MGL, being equal (III. 16. El.) to MIH in the same segment, is equal (I. 22. El.) to the exterior angle MEL, and consequently (III. 17. cor. 2. El.) the quadrilateral figure MGEL being thus contained in a circle, the angle LME is equal to LGE. Draw LN making the angle MLN equal to EGO, and (I. 30. El.) the exterior angle LNE will be equal to CGO. But the triangles GOC and HOC are obviously equal, and, therefore, the angle CGO = CHO = CLE = EKI. Whence the triangles IEK and LEN are similar, and IE : KE :: EL : EN, and consequently, (V. 6. El.) KE.EL = IE.EN.



Make, therefore, PQ to PR, as EN to EM. The ratio of AE.ED to BE.EC is hence compounded of that of PR to PQ; and of PQ to ST, or it is the same with the ratio of PR to ST. But as the point of section E approaches to O, the angle EGO, or MLN, evidently diminishes, and consequently the point N, in a corresponding degree, approximates to M. Hence the extreme term which PR can never pass, is PQ; and therefore the limiting ratio of the rectangle AE, ED to BE, EC is that of PQ to ST, or of the square of IO to the square of BO.

The point O of ultimate section is hence easily determined. Because BI and CH are parallel, BI : CH : : BO : OC, and BI^2 or AB.BD : CH^2 or AC.CD : : BO^2 : OC^2. Wherefore BC must be divided (I. 24. Anal.) into segments BO and OC, which are in the subduplicate ratio of the rectangle AB, BD to AC, CD.

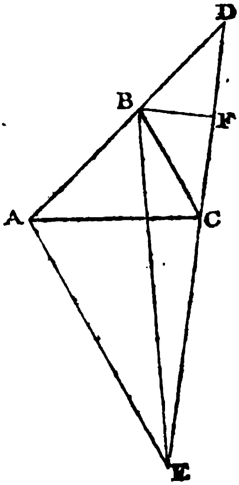
But the limiting ratio may be found in a more direct manner. For join IG, and draw DV perpendicular to it, join DG, DI, CV, and BV, and, having drawn the diameter IZ, join GZ. Because the angles DGC and DIV stand upon equal arcs DH and DG, they are equal (III. 17. El.); but the quadrilateral figures DCGV and DBIV, being right angled at B, at C and V, are each contained in a circle (III. 17. cor. 1. El.); wherefore (III. 16. El.) the angle DGC is equal to DVC, and the angle DIV to DBV, and consequently the angles DVC and DBV are equal. Hence the triangles CDV and VDB, having besides a common vertical angle, are similar; and, therefore, BD:DV::DV:DC, and (V. 6. El.) BD.DC=DV^2. But (VI. 15. cor. El.) DG^2=AD.DC, and consequently DG^2—DV^2 or (II. 10. El.) GV^2=AD.DC—BD.DC, or AB.DC. In the same manner, it is shown that IV^2= AC.DB. Whence IG is given, being the difference between the sides of two squares that are equal to the rectangles AC, DB, and AB, DC. Again, the angle BIO, being equal to the alternate angle GHI, is equal (III. 16. El.) to GZI, and the right angle OBI is equal to the angle IGZ in a semicircle; wherefore the triangles IOB and ZIG are similar, and IO : BO : : IZ or AD : IG. Hence the limiting ratio of AE.ED to BE.EC, or that which marks the state of a *minimum*, is the duplicate ratio of AD to the difference of the sides of squares that are equal respectively to the rectangle AC, DB and to the rectangle AB, DC.

PROP. XIX. THEOR.

If a triangle have a given angle, the excess of the square of the sum of the containing sides above the square of the base, has a given ratio to the area of the triangle.

Let ABC be a triangle, in which AB is produced till BD be equal to BC; the excess of the square of AD above the square of AC, has a given ratio to the area of the triangle.

ANALYSIS.



Draw AE parallel to BC, and meeting DC produced in E, from B let fall the perpendicular BF, and join BE.

The triangle CBD being isosceles, the angle CDB (I. 10. El.) is equal to DCB, but (I. 22. El.) DCB is equal to CEA; hence the angles EDA and DEA are equal, and the triangle DAE is isosceles. Wherefore (II. 20. El.) $AD^2 = AC^2 + DC.CE$, or $AD^2 - AC^2 = DC.CE$. Again, because AE is parallel to BC, the triangle ABC has (II. 1. El.) the same area as EBC, or (II. 6. El.) is half the rectangle BF.CE. Consequently the excess of the square of AD above the square of AC, is to the area of the triangle ABC, as DC.CE to ½BF.CE, that is, (V. 23. cor. 2. El.) as DC to ½BF, or (V. 3. El.) as 4DF to BF. But the given angle ABC, being (I. 32.) equal to the two angles CDB and BCD, is double of either, and thus the angle BDF is given; whence the right-angled triangle DFB is given in species, and therefore the ratio of DF to BF is given. It thence follows, that the ratio of 4DF to BF, or that of the excess of the square of AD above the square of AC to the area of the triangle ABC, is given.

COMPOSITION.

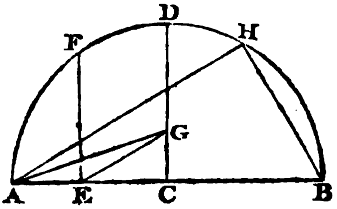
The same construction remaining, DC.CE : BF.CE : : DC : BF; but $DC.CE = AD^2 - AC^2$, and BF.CE is double of the triangle ABC; whence 2DC is to BF, as the excess of the square of AD or that of the sum of the sides AB and BC above the square of the base AC, to the area of the triangle ABC.

PROP. XX. PROB.

If, from the extremity of the diameter of a circle, a straight line be drawn to a point in the perpendicular radius, such that triple its square be equal to the square of a perpendicular from the circumference and the squares of the segments into which the diameter is thus divided; the straight line that joins the points of section and of termination will make a given angle with the diameter.

In the semicircle ADB, EF and the radius CD being at right angles to AB, and AG drawn so that $3AG^2 = AE^2 + EF^2 + EB^2$; if EG be joined, the angle CEG is given.

ANALYSIS.



For join CF. Because AB is bisected in C, $AE^2 + EB^2 = 2AC^2 + 2EC^2$ (II. 18. cor. El.) and consequently, from the hypothesis, $3AG^2 = 2AC^2 + 2EC^2 + EF^2$; but (II. 10. El.) $EC^2 + EF^2 = CF^2$, or AC^2, and hence $3AG^2 = 3AC^2 + EC^2$. Again, $AG^2 = AC^2 + CG^2$, or $3AG^2 = 3AC^2 + 3CG^2$; wherefore $EC^2 = 3CG^2$, or $EG^2 = 4CG^2$ and $EG = 2CG$. The ratio of EG to CG, and the right angle at C being thus given, the triangle EGC is (VI. 14. El.) given in species, and consequently the angle CEG is given.

COMPOSITION.

Inflect BH equal to the radius of the circle, join AH, draw EG parallel to it meeting CD in G, and join AG; then $3AG^2=AE^2+EF^2+EB^2$.

For join CF. The triangles AHB and ECG being evidently similar, AB : BH : : EG : CG; but AB=2BH, and therefore (V. 5. El.) EG = 2CG. Whence $EG^2 = 4CG^2$, and $EC^2=3CG^2$; consequently $3AG^2=3AC^2+3CG^2=3AC^2+EC^2=2AC^2+2EC^2+AC^2-EC^2$. Now $2AC^2+2EC^2=AE^2+EB^2$, and $AC^2-EC^2=EF^2$; wherefore $3AG^2=AE^2+EF^2+EB^2$.

PROP. XXI. PROB.

To find a point in the diameter of a circle, such that the square of a straight line inflected from it at a given angle to the circumference, shall have a given ratio to the rectangle under the segments of the diameter.

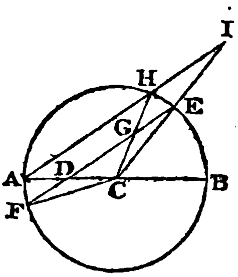
Let it be required, from the diameter AD, to draw DE at a given angle with it, and such that the square of DE shall have a given ratio to the rectangle under its segments AD, DB.

ANALYSIS.

Produce ED to the circumference in F. Make EG = FD, join CF, draw the radius CGH, join AH, and produce it to meet the extension of CE in I.

Because CE is equal to CF, the angle CEF is (I. 10. El.) equal to CFE. Wherefore the triangles CGE and CDF, having thus the angle CEG equal to CFD, and the sides CE and EG equal to CF and FD,—are (I. 3. El.) equal, and consequently the angle ECG is equal to FCD; whence (III. 12. El.) the arc HE is equal to AF, and therefore (III. 18. cor. El.) AH is parallel to DE. But the angle

BDE is given, and thence BAH; wherefore the chord AH is given. Again, the rectangle AD, DB, being equal to FD.DE (III. 26. El.), is also equal to DE.EG; and therefore DE^2 is to DE.EG, or (V. 25. cor. 2. El.) DE is to EG, in the given ratio; but (VI. 2. El.), DE : EG : : AI : IH, consequently AI is to IH in a given ratio, and hence AH is to HI likewise in a given ratio. Wherefore since AH is given, IH and the point I are given; and thence IC, the point E, and DE are all given.



COMPOSITION.

Draw AH at an inclination with AB equal to the given angle, and produce it to I, so that AI shall be to IH in the given ratio, join IC, and draw ED parallel to IA; D is the point required.

For produce ED to meet the circumference in F, and join CF and CGH. Because AI : IH : : DE : EG, (VI. 2. El.) DE has to EG the given ratio, and consequently DE^2 has to DE.EG the same ratio. But FE being parallel to AH, the arc HE (III. 18. El.) is equal to AF, and thence the angle HCE is equal to ACF; the triangles CGE and CDF, having thus the side CE equal to CF, and the angles ECG and CEG equal to FCD and CFD,—are (I. 20. El.) equal, and hence the side EG is equal to FD. Wherefore DE.EG = DE.FD = AD.DB, and consequently DE^2 is to AD.DB in the given ratio.

Cor. It is obvious that DE^2 cannot be equal to AD.DB, unless the point H coincide with A; in which case, AI becomes a tangent, and DE a perpendicular to the diameter, and the proposition merges in (III. 26. cor. 1. El.)

PROP. XXII. PROB.

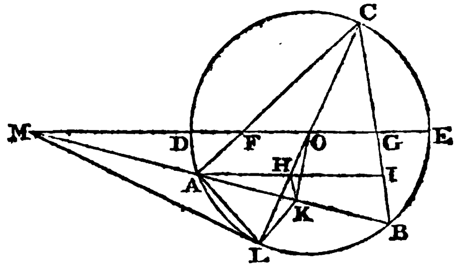
From two given points in the circumference of a given circle, to inflect straight lines to another point in the opposite circumference, such as to intercept, on either side of the centre, equal segments of a given diameter.

Let it be required, from the points A and B, to inflect AC and BC, so as to intercept, on the diameter DE, equal portions from the centre.

ANALYSIS.

Join BA, and produce it and the diameter ED to meet in M, draw COL, from O let fall the perpendicular OK upon AB, join LK, through A draw AHI parallel to DE, and join HK.

The parallels FG and AI are cut proportionally by the diverging lines CA, CH, and CI (VI. 1. El.); but FO is equal to OG, and consequently AH is equal to HI. Wherefore (II. 4. El.) HK is parallel to IB, and the angle AKH is equal to ABI (I. 22. El.); and since the angle ABI or ABC is equal to ALC (III. 18. El.), the angle AKH is equal to ALC or ALH, and hence (III. 18. cor. El.) the quadrilateral figure AHKL is contained in a circle. Consequently (III. 16. El.) the angle HAK is equal to HLK; but HAK is equal (I. 22. El.) to OMK,



which is therefore equal to HLK or OLK, and thence the quadrilateral figure MOKL is also contained in a circle. Wherefore (III. 16. El.) the angle MLO is equal to MKO; but MKO is a right angle, and consequently MLO is likewise a right angle, and thence (III. 20. El.) ML is a tangent. But the point M, being the concourse of ED and BA, is given, and therefore the tangent ML to the given circle is given (III. 22. El.); whence the diameter LC, and the point C are given.

COMPOSITION.

Produce ED and BA to meet in M, draw the tangent ML and the diameter LC; the straight lines AC and BC will cut off from the centre equal portions, OF and OG, of the given diameter ED.

For draw AI parallel to DE, and OK perpendicular to AB, and join LK and KH.

Because ML is a tangent, MLO is a right angle, and, therefore, equal to MKO; consequently (III. 16. El.) MKL is equal to MOL, that is, (I. 22. El.) to AHL: Wherefore the quadrilateral figure AHKL is contained in a circle, and hence (III. 16. El.) the angle ALH is equal to AKH; but, for the same reason, ALH or ALC is equal to ABC or ABI, and consequently AKH is equal to ABI, and (I. 22. El.) KH parallel to BI. Now since AK is equal to KB, it follows that AH is equal to HI, and hence that FO is equal to OG.

PROP. XXIII. PROB.

Two straight lines being given, to draw, through a given point, another straight line, cutting off segments which are together equal to a given straight line.

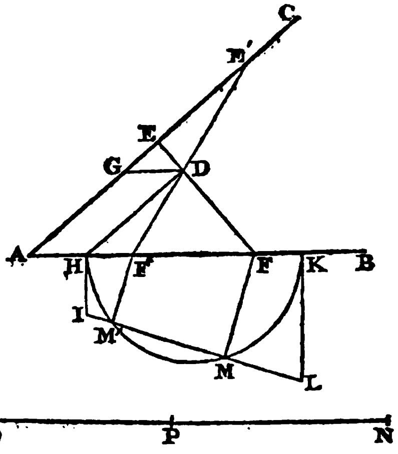
Let AB, AC be two straight lines, and D a given point, through which it is required to draw a straight line EF, so as to cut off the segments AE and AF, that are together equal to ON.

The point D may lie either within or without the angle formed by the straight lines AB and AC.

1. *Let D have an internal position.*

ANALYSIS.

Draw DG and DH (I. 23. El.) parallel to AB and AC. Because the point D is given, and AB, AC are given in position, the parallelogram AGHD is given. And since the triangles EDG and DFH are evidently similar, $EG : GD :: DH : HF$, and therefore $EG.HF = GD.DH$. But AG and AH, or DH and GD, being given, the rectangle GD, DH is given, and therefore EG, HF is given. Make $FK = EG$, and the rectangle HF, FK is hence given; but HK, being equal to HF and FK or the excess of AF and AE above GD and DH, is given, and consequently (VI. 17. El.) its segments HF, FK are given; whence the point H being given, the point of section F or F′, and the straight line EDF or E′DF′, are given.



COMPOSITION.

Draw the parallels DG and DH. From ON, the sum of the two segments AE and AF, cut off OP=AG+AH, and make HK=PN. On HK describe a semicircle, from the extremities of the diameter erect the perpendiculars HI and KL equal to AH and AG, join IL, and at right angles to this, and from the point or points where it meets the circumference, draw MF or M′F′; EDF or E′DF′ is the straight line required.

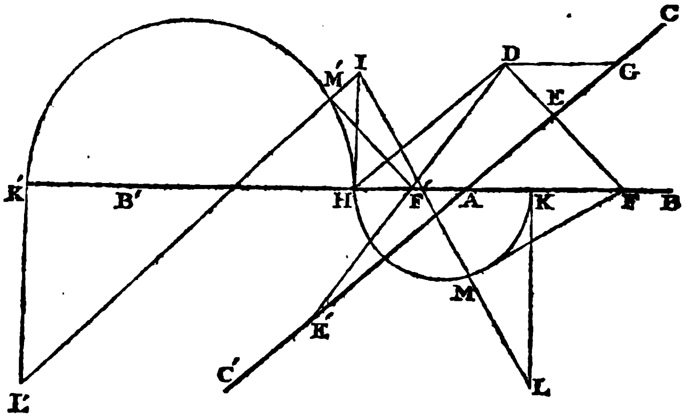
For (VI. 17. El.) HI.KL=HF.FK, and consequently AH.AG=HF.FK. But, from the similar triangles EGD and DHF, EG : GD or AH : : DH or AG : HF, and therefore (V. 6. El.) AH.AG=HF.EG; whence HF.FK =HF.EG, and FK = EG. And since AG + AH = OP, and HF+EG=HK=PN, it follows that AG+EG +AH+HF, or AE+AF=ON.

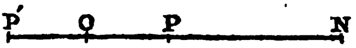
2. *Let the point D have an external position with respect to the straight lines AB and AC.*

ANALYSIS.

Draw DG parallel to AB, and DH parallel to AC and meeting AB produced. The triangles EDG and DHF being similar, EG : DG : : DH : HF, and (V. 6. El.) EG.HF=DG.DH; but DG and DH are both given, and hence the rectangle under EG and HF is given. Make FK=EG, and therefore HK=HF—EG=DG+AF—(DH—AE)=AF+AE—(DH—DG); whence HK and the rectangle HF, FK are given, and consequently (VI. 17. El.) the point F is given.

If DF′E′ intersect the straight lines AB and AC on the other side of their vertex A, the triangles E′DG and DF′H are still similar, and E′G : DG : : DH : HF′; wherefore E′G.HF′, being equal to DG.DH, is given. Make F′K′ = E′G, and thence HK′ = E′G—HF′ = AE′+DH—(DG—AF′) = AF′ + AE′ + (DH—DG); consequently





HK′ and the rectangle HF′.F′K′ are given, and therefore (VI. 17. El.) the point F is given.

COMPOSITION.

Make OP or OP′ equal to the difference of the parallels DH and DG, from H place likewise towards opposite parts $HK = PN$ and $HK' = P'N$, on HK and HK′ describe semicircles, from H erect the perpendicular HI equal to DG, and, from K and K′, the perpendiculars KL and K′L′, each equal to DH, join IL and IL′, and at right angles to these, from the points of section M and M′, draw MF and M′F′; the straight lines DEF and DF′E′ will cut off segments from AB and AC, which are together equal to ON.

For (VI. 17. El.) $HF.FK = HI.KL = DG.DH$; but $DG.DH = HF.EG$, and consequently $HF.EG = HF.FK$, or $EG = FK$. Wherefore $HK = HF - EG = AF + AE - (DH - DG)$; and since $HK = PN = ON - (DH - DG)$, it follows that $AF + AE = ON$.

In like manner, it is shown that $E'G = F'K'$, and hence $HK' = E'G - HF' = AF' + AE' + (DH - DG)$; but $HK' =$

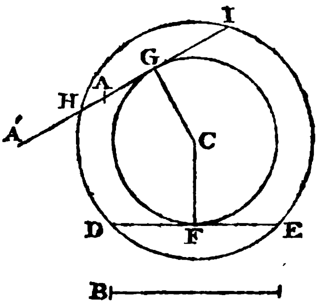
$P'N'=ON+(DH-DG)$, and consequently $AF'+AE'=ON$.

PROP. XXIV. PROB.

Through a given point, to draw a straight line, so that the part intercepted by the circumference of a given circle, shall be equal to a given straight line.

Let A be a point, through which it is required to draw a straight line HI, limited by a given circumference and equal to B.

ANALYSIS.



Take any point D in the given circumference, and inflect DE equal to B. Because DE is equal to B, it is equal to HI, and, therefore, (III. 10. El.) the chords HI, DE are equally distant from the centre of the circle, or CG=CF. But DE being given, CF is given, and thence the circle described from C through F and G; wherefore the point A being given, the tangent AG to that circle is given, and consequently HI is given in position.

COMPOSITION.

Inflect DE equal to B, from C let fall the perpendicular CF, with which distance describe an interior concentric circle, and draw (III. 22. El.) the tangent HAI.

It is evident that the chords HI and DE, being equidistant from the centre, are both of them equal to B.

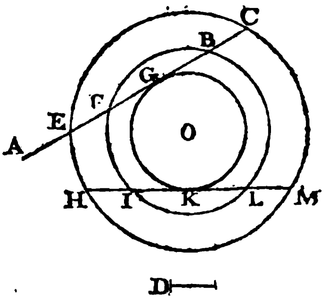
PROP. XXV. PROB.

Through a given point, to draw a straight line, such that the part of it intercepted between two concentric circles shall be equal to a given straight line.

Let it be required, through the point A, to draw the straight line ABC, so that the part BC intercepted by the two concentric circles HECM and IFBL shall be equal to D.

ANALYSIS.

From any point H, in one of the circumferences, inflect HM=EC, and upon these let fall the perpendiculars OK and OG. The equal chords HM and EC are therefore (III. 10. El.) equidistant from the centre, and reciprocally IL is equal to FB; consequently the halves of these are equal, or HK = GC, and IK=GB; whence the difference HI, being equal to BC, is given. But since the point H is given, the point I and the chord HM are given; and the circle which touches at K being given, the tangent AGC is also given.



COMPOSITION.

In the circumference of one of the circles, having assumed a point H, insert HI equal to D, and produce it to M, upon this let fall the perpendicular OK, with which as a radius describe a circle, and apply it to the tangent ABC; then will the intercepted portion BC be equal to D.

For the chords EC and FB are (III. 10. El.) equal to the equidistant chords HM and IL; consequently their halves are equal, or GB=IK, and GC=HK, and hence BC=HI=D.

It is evident, that the interval BC between the concentric circles will be least when AC passes through the centre, and greatest when it touches the inner circle. Wherefore D is limited on both sides; not being less than the difference of the radii of the circles, nor its square greater than the difference of their squares.

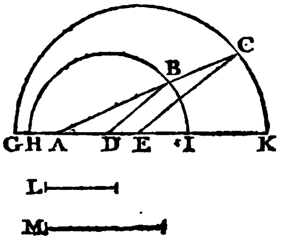
PROP. XXVI. PROB.

Two circles described upon the same straight line being given, to draw from a point similarly placed in it another straight line, so that the part intercepted by the circumferences shall be equal to a given straight line.

Let D, E be the centres of the two circles, and let AD:AE::DI:EK; it is required from A to draw ABC, such that BC shall be equal to L.

ANALYSIS.

Join BD and CE. Because AD : AE : : DI or DB : EK, or EC, therefore (VI. 1. cor. El.) DB is parallel to EC; whence AD : DE : : AB : BC, and since AD and DE are given, the ratio of AB to BC is given; but BC is given, and consequently AB is given, both in magnitude and position.



COMPOSITION.

Make (VI. 3. El.) EK—DI : DI : : L : M, and from A inflect AB equal to M; ABC is the straight line required.

For since, by hypothesis, AD : AE : : DI or DB : EK or EC, DB is parallel to EC; wherefore DB or DI : EC or EK : : AB : AC, and consequently (V. 11. cor. El.) EK—DI : DI : : BC : AB; but EK—DI : DI : : L : M or AB, whence BC : AB : : L : AB, and therefore (V. def. 10. El.) BC = L.

PROP. XXVII. PROB.

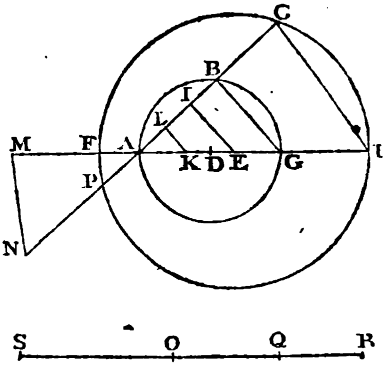
Two circles described upon the same straight line being given, to draw, from the extremity of either diameter, another straight line, so that the part of it intercepted by the circumferences shall be equal to a given straight line.

Let it be required to draw ABC, so that the intercepted portion BC shall be equal to QR.

ANALYSIS.

Join BG, CH, and FP, from E, the centre of the exterior circle, let fall upon AC the perpendicular EI, cut off IL = IB and draw LK parallel to BG, in the extension of AH make (VI. 3. El.) AK : AG : : AF : AM, and, from the point M, draw MN parallel to FP, and meeting the production of AC.

Because LK is parallel to BG and FP to MN, therefore (VI. 1. El.) AK : AG : : AL : AB, and AF : AM : : AP : AN; but, by construction, AK : AG : : AF : AM, and



consequently, AK : AG : : AL : AB : : AP : AN. Whence (V. 19. El.) AK : AG : : AL + AP or PL : AB + AN or BN. Now, since (III. 4. El.) IP = IC, and IL = IB, therefore PL = BC or QR; and LK, IE, and BG being parallel lines, KE = EG (VI. 1. El.) and thence AK is given; wherefore three terms of the analogy being given, the fourth term BN is given, and consequently BN + BC, or NC, is given. But the angle ACH is equal to AFP (III. 16. El.) which again (I. 22. El.) is equal to AMN, and hence the triangles CAH and ANM, having also the same vertical angle, are similar, consequently AH : AC : : AN : AM, and (V. 6. El.) AH.AM = AC.AN, wherefore NC and the rectangle under its segments AC, AN being given, AC is given in magnitude (VI. 17. El.) and hence likewise in position.

COMPOSITION.

Having cut off KE = EG, make (VI. 3. El.) AK : AG : : AF : AM, and AK : AG : : QR : QS, divide (VI. 19. El.) SR in O, such that SO.OR = AH.AM, and inflect AC = OR; AC is the straight line required.

For join CH, BG, FP, and draw MN parallel to FP, and EI and KL parallel to CH. Since IL = IB and IP = IC, therefore PL = BC. The triangles CAH and ANM are similar, AH : AC : : AN : AM, and AH.AM = AC.AN;

but, by construction, AH.AM=SO.OR, and AC=OR, consequently AN=SO. Now, from the property of parallels, AK : AG : : AL : AB, and, by hypothesis, AK : AG : : AF : AM, or AP : AN; wherefore (V. 19. El.) AK: AG : : AL+AP or BC : AN+AB or BN. Whence BC : BN : : QR : QS, and (V. 11. El.) BC: CN : : QR : SR; but CN=SR, and consequently BC=QR.

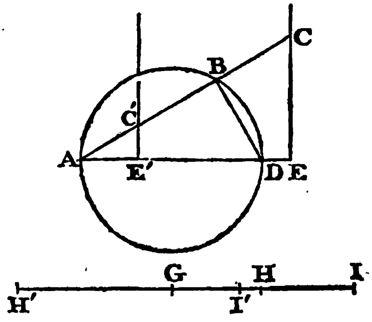
PROP. XXVIII. PROB.

From the extremity of the diameter of a given circle, to draw a straight line, so that the part of it intercepted between a given perpendicular and the circumference, shall be equal to a given straight line.

Let it be required from A to draw AC, such that the intercepted portion BC shall be equal to GH.

ANALYSIS.

Join BD. The angle ABD, being in a semicircle, is a right angle, and therefore equal to AEC; consequently the triangles DAB and CAE, having besides a common angle at A, are similar, and AB : AD : : AE : AC, and hence AB.AC=AD.AE. But the rectangle AD, AE is given, and thence AB, AC; and since BC is given in



magnitude, therefore (VI. 17. El.) AB is given in magnitude, and consequently in position.

COMPOSITION.

Produce GH (VI. 17. El.) till GI.IH = AD.AE, and inflect IH from A to B; AB is the straight line required. For join BD. The triangles ABD and AEC being evidently similar, AB : AD : : AE : AC, and consequently AB.AC = AD.AE = GI.IH; but AB = IH, whence AC = GI, and therefore BC = GH.

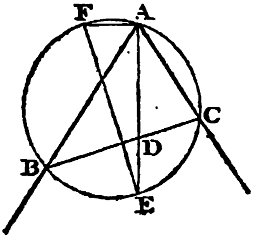
PROP. XXIX. PROB.

Through a given point in the line bisecting a given angle, to draw a straight line limited by the sides, and equal to a given straight line.

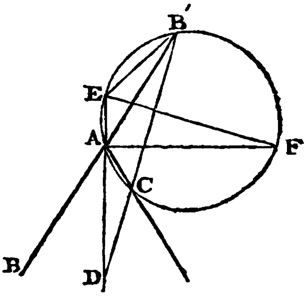
Let it be required, through the point D, situate in the straight line AD which bisects the angle BAC, to draw BC equal to a given straight line.

ANALYSIS.

About the points B, A and C, describe (III. 9. cor. El.) a circle, draw the diameter EF, and join AF. Because BC and the angle BAC are given, the circumscribing circle (III. 23. El.) and consequently the triangle BAC, are given in magnitude: But since the angle BAE is equal



to CAE, the arc BE is (III. 16. cor. El.) equal to CE; and hence the chord BC is bisected at right angles by the diameter EF. Wherefore AD being given, AE is, by the last proposition, given in magnitude, and thence DB is given in magnitude, and consequently in position.



COMPOSITION.

On the given straight line describe (III. 23. El.) a segment BAC, containing an angle equal to the given angle, and complete the circle, bisect the arc BAC in E, and from that point draw, by the last proposition, EAD, such that AD shall be equal to the distance of the given point from the vertex; and DB, DC are the segments of the required line, from which its position is immediately determined.

For the angle BAC is equal to the given angle, and AD bisects it, since the arc BE$=$CE; but AD is besides equal to the distance of the given point from the vertex, and BC is equal to the given straight line. Wherefore all the points and lines retain, by this construction, their relative position.

Cor. Since AE cannot exceed the diameter FE, the limiting case will occur when these lines coincide; wherefore BC is the least possible when it is at right angles to AD, and consequently intercepts equal segments AB and AC.

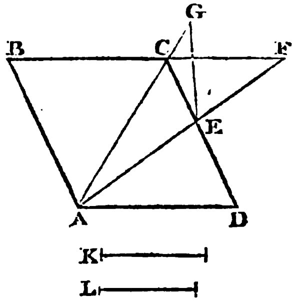
PROP. XXX. PROB.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner.

Let ABCD be a rhombus, of which the side BC is produced; it is required, from the opposite corner A, to draw AEF, such that the exterior portion EF shall be equal to a given straight line.

ANALYSIS.

Join AC, and, meeting this produced, draw EG, making the angle AEG equal to ACF. The triangles CAF and EAG are evidently similar, and AC : CF : : AE : EG; but CE being parallel to AB, BC : CF : : AE : EF (VI. 1. El.); whence (V. 17. El.) AC : BC : : EF : EG. But AC, BC, and EF being given, EG is (VI. 3. El.) also given. Again, the angle ACD is (I. 2. El.) equal to ACB, and therefore to FCG; consequently adding ECF. to each, the whole angle ACF, or AEG, is equal to ECG. Hence the triangles AGE and EGC are similar, and AG : EG : : EG : GC, or $AG.GC = EG^2$. Wherefore the rectangle AG, GC is given, and consequently (VI. 17. El.) the point G, and thence the point E and the straight line AF.



COMPOSITION.

Let the intercepted segment be equal to K, join AC, make AC : BC : : K : L, divide AC in G (VI. 17. El.) so that AG.GC$=$L^2, and from G, with the radius L, describe a circle cutting CD in E ; AEF is the straight line required.

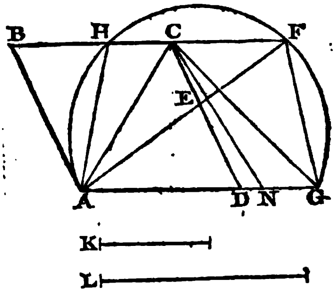
For since AG.GC$=$L^2 $=$ EG^2, AG : EG : : EG : GC, and therefore the triangles AGE and EGC are similar, and the angle AEG is equal to ECG, or ACF; whence the triangles AFC and AGE are likewise similar, and AC : CF : : AE : EG ; but (VI. 1. El.) BC : CF : : AE : EF, and consequently (V. 17. El.) AC : BC : : EF : EG. Now AC : BC : : K : L or EG ; wherefore EF$=$K.

Otherwise thus.

ANALYSIS.

Draw FG making the angle AFG equal to ADC, cut off CH$=$CE, from C inflect CN$=$CA, and join CG and AH.

The triangle ACN being isosceles, the angle CAN is (I. 10. El.) equal to CNA ; and since the diagonal AC bisects the angle BCD of the rhombus, the triangles ACE and ACH are (I. 3. El.) likewise equal, and hence AE is equal to AH, and the angle CAE equal to CAH. And because the triangles ADE and AFG are similar, AD : AE : : AF : AG and AD.AG$=$AE.AF. But the angle ACD, being equal to CAD, is equal to CNA, and consequently the triangles ADC and ACN are similar ; whence AN : AC : : AC : AD, and therefore AN.AD $=$AC^2. Again, because AC bisects the vertical angle HAF (VI. 17. El.) FA.AH $=$ AC^2 $+$ FC.CH, that is, FA.AE$=$AC^2$+$ FC.CE ; wherefore FC.CE $=$ FA.AE $-$ AC^2, that is, AG.AD—AN.AD, or NG.AD. But BA and CE being parallel, FC : EF : : AD : AE : : AF : AG, and CE : EF : : AB or AD : AF ; consequently (V. 22. El.) FC.CE : EF^2 : : AD : AG : : (V. 25. cor. 2. El.) NG.AD : NG.AG ; since therefore FC.CE = NG.AD, it follows (V. 8. and 4. El.) that EF^2 = NG.AG. Hence (VI. 17. El.) AG and the point G are given, and the angle AFG, being equal to ADC, is (III. 23. El.) contained in a given segment of a circle ; wherefore the intersection F and the inflected line AF, are given.



COMPOSITION.

Let K be equal to the intercepted portion of the straight line which is to be inflected from A, and find (II. 12. El.) L the side of a square equivalent to the squares of K and of the diagonal AC, produce AD, and from C place CG equal to L, upon AG describe (III. 23. El.) a segment of a circle containing an angle equal to ADC, and join A with the point of intersection F ; AF is the straight line required.

For inflect CN = CA, and join GF and AH.

The triangles AHC and AEC are equal ; for the angle AFG, being by construction equal to ADC, is equal (I. 22. El.) to the alternate angle formed by the production of BA with AD, and consequently (III. 21. cor. El.) AB touches the circle at A ; whence the angle BAH = HFA = DAE, and taking these from the equal angles BAC and DAC, there remains CAH = CAE, but the angles ACH and ACE are also equal, and the side AC is common to the two triangles; wherefore $AH=AE$, and $CH=CE$. And because the triangles ADE and AFG are similar, AD : AE : : AF : AG, and $AD.AG=AE.AF$. Again, the triangles ANC and ACD being similar, AN : AC : : AC : AD, and $AN.AD=AC^2$. But FC : EF : : AD : AE : : AF : AG, and CE : EF : : AB or AD : AF; consequently $FC.CE : EF^2$: : AD : AG : : NG.AD : NG.AG; and since AC bisects the angle FAH, $FC.CH+AC^2=FA.AH=FA.AE=AG.AD=AN.AD+NG.AD$, it follows that FC.CH, or $FC.CE=NG.AD$, and hence $EF^2=NG.AG$. Now $K^2=CG^2-AC^2=$ (II. 17. cor.) NG.AG; wherefore $EF^2=K^2$, and $EF=K$.

PROP. XXXI. PROB.

Through two given points, to describe a circle touching a straight line given in position.

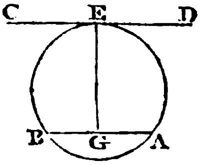
Let it be required to describe a circle through the points A, B, and touching the straight line CD.

It is evident that CD must either be parallel or inclined to the straight line which joins the points A and B.

1. *Let CD be parallel to AB.*

ANALYSIS.

From the point of contact E, draw (I. 6. El.) EG perpendicular to CD. Hence (III. 20. cor. El.) EG passes through the centre of the circle, and since it is also perpendicular to AB (I. 22. El.) it bisects that chord at right angles (III. 4. El.) the point G is therefore given,



and the perpendicular GE; consequently the three points A, E, and B being thus given, the circle AEB is given.

COMPOSITION.

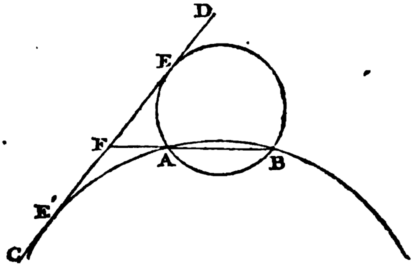
Draw (I. 7. El.) GE bisecting AB at right angles, and (III. 9. cor. El.) through the points, A, E and B describe a circle; this will touch the straight line CD.

For (III. 6. El.) GE must pass through the centre of the circle, and (I. 22. El.) it meets the parallels CD and AB at right angles; whence (III. 20. El.) CD is a tangent to the circle.

2. *Let CD be inclined to AB.*

ANALYSIS.

Produce BA to meet CD in F. Then (III. 26. cor. 2. El.) $FE^2 = AF.FB$; but the point of concourse being given, the rectangle AF, FB is given, and consequently FE and the point E. Wherefore since the three points A, E, and B are given, the circle AEB is given.



COMPOSITION.

Produce BA to meet CD in F, find (VI. 16. El.) FE or FE′ a mean proportional to AF and FB, and (III. 10. cor. El.) through the points A, B, and E, or A, B, and E′, describe a circle; this will touch the straight line CD.

For since AF : FE : : FE : FB, therefore (V. 6. El.)

$FE^2 = AF.FB$, and consequently (III. 28. El.) FE, or FE′, touches the circle.

PROP. XXXII. PROB.

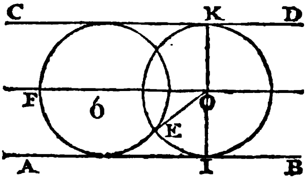
Through a given point, to describe a circle touching two straight lines given in position.

Let it be required, through the point E, to describe a circle touching AB and CD.

1. *When AB is parallel to CD.*

ANALYSIS.

Through the centre O draw the parallel FO and the common perpendicular KI. It is evident that the radius OI is given, and consequently FO is given in position; but OE, being equal to OI, is given, and therefore the centre O is given.



COMPOSITION.

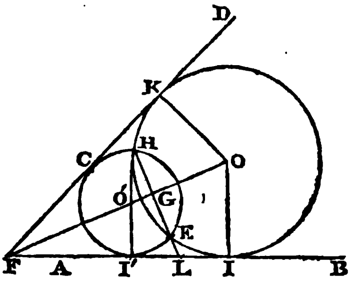
Draw a parallel FO bisecting the distance between the straight lines AB and CD, and from E with a radius equal to half that distance intersect FO in O, or O′; this point is the centre of the circle required. For $OE = OI = OK$, and the circle which passes from E must touch at K and I.

2. *When CD is inclined to AB.*

ANALYSIS.

Produce BA and DC to meet in F, join OI, OK, and OF, and from E draw EGH perpendicular to OF.

The triangles OKF and OIF, being (III. 20. El.) right-angled, and having the side OK equal to OI and the side OF common, are (I. 22. El.) equal, and consequently the angle OFK is equal to OFI; wherefore since the point of concourse F is given, the straight line OF is given. But, the point E being given, the perpendicular EH is thence given, and (III. 4. El.) GH being equal to GE, the opposite point H is given. Two points E, H, and a straight line AB, are thus given, and therefore, by the last proposition, the circle EHKI is given.



COMPOSITION.

Produce BA and DC to meet in F, draw (I. 5. El.) FO bisecting the angle BFD, from E (I. 6. El.) let fall the perpendicular EG, and extend it both ways, making GH =GE, find (VI. 16. El.) LI, or LI′, a mean proportional to HL and LE, and through the points H, E, I, or H, E, I′, describe a circle; this circle will touch both the straight lines AB and CD.

For the centre of the circle which passes through E and H, must (III. 5. El.) occur in FO; let it be O, join OI and draw the perpendicular OK. Because $HL.LE=LI^2$, the circle touches AB at I, and hence OIF is a right angle; consequently the triangles KOF and IOF having the angles OKF and OFK equal to OIF and OFI, and the side OF common, are (I. 20. El.) equal, and therefore OI = OK; whence the circle described from O passes through K, and (III. 20. El.) must touch CD at that point.

Cor. If the given point E should fall on AB, and thus coincide with the point of contact,—the problem will become much simpler; for the centre O, lying in the intermediate or bisecting line FO, will be determined by the intersection of the perpendicular IO.

PROP. XXXIII. PROB.

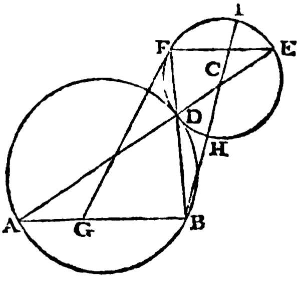
Through two given points, to describe a circle touching a given circle.

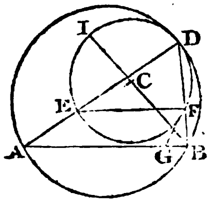
Let it be required, through the points A and B, to describe a circle, touching another circle whose centre is C.

ANALYSIS.

Through D, the point of contact, draw ADE and BDF, join EF, at F (I. 5. cor. 2. El.) apply the tangent FG, and draw BHCI.

Because FG touches the given circle, the angle BFG is (III. 21. El.) equal to FED, and therefore equal to BAD, since (III. 24. El.) FE and AB are parallel; but the triangles BGF and BDA have likewise a common angle at B, and are hence similar; therefore BF : BG : : BA : BD, and (V. 6. El.) BA.BG = BF.BD = (III. 26. El.) BI.BH. But BI and BH are given, and thence the





rectangle BA, BG is given, and consequently (II. 8. El.) the point G is given. Hence the tangent GF, and D, the intersection of BF, are given; wherefore the circle that passes through the three points A, D, and B is given.

COMPOSITION.

Make (VI. 3. El.) BA:BI :: BH : BG, draw (III. 20. El.) the tangent GF, join BF cutting the given circumference in D, and (III. 9. cor. El.), through the points A, D, and B, describe a circle; this will touch the circle FDE.

For draw ADE, join FE, and draw BHCI. Since BA : BI :: BH : BG, therefore (V. 6. El.) BA.BG = BI.BH = (III. 26. El.) BF.BD; whence BF : BG :: BA : BD, and consequently the triangles BGF and BDA, having the same vertical angle, are (VI. 13. El.) similar, and hence the angle BFG is equal to BAD. But (III. 21. El.) BFG is equal to FED, and thus the alternate angles BAE and FEA are equal, and FE is parallel to AB; whence (III. 24. El.) the two circles touch at D.

It is obvious, that the tangent GF will have two positions, and consequently the line BDF may cut the given circle in two points. The circle which passes through AB and D may therefore, in each case, have an exterior as well as an interior position, though here omitted, to avoid confusing the figure.

PROP. XXXIV. PROB.

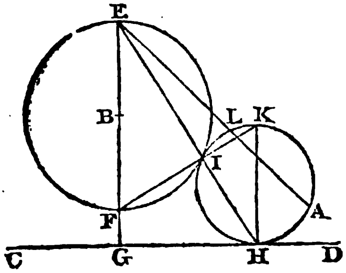
Through a given point, to describe a circle, touching a given circle and a straight line which is given in position.

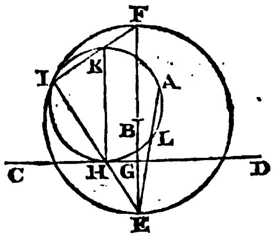
Let it be required, through the point A, to describe a circle touching the straight line CD and the circle whose centre is B.

ANALYSIS.

From the centre of the given circle let fall the perpendicular EBG, join EI and extend it to H in the straight line CD, also draw FIK and join HK.

The angle HIK, being equal to EIF which stands in a semicircle, is (III. 19. El.) a right angle, and consequently HK is the diameter of the circle ILA, and H the point of contact. The triangles HEG and FEI are therefore similar, HE : EG :: EF : EI, whence HE.EI = EG.EF. Join ELA and (III. 26. El.) AE.EL = HE.EI or EG.EF; but the rectangle EG, EF is given, and consequently that of AE, EL; now AE being given, the point L is hence given. Wherefore, since the two points A, L, and the straight line CD, are all given,—the circle HIA is given.





COMPOSITION.

Join EA, draw the perpendicular EG, make (VI. 3. El.) AE : EG : : EF : EL, and by Prop. 31. of this Book, describe a circle through the points A, L, and touching the straight line CD; this circle will also touch the given circle.

For draw the diameter HK, join EH cutting the circumference EIF, and draw FIK meeting HK.

The triangles HEG and FEI being evidently similar, HE: EG : : EF : EI, and HE.EI=EG.EF; but AE : EG : : EF : EL, and AE.EL=EG.EF; wherefore HE.EI=AE.EL, and (III. 28. El.) the point I must lie in the circumference HIK. But the two circles also touch in I; for EG being parallel to HK, the angles IEF and IHK are equal, which are again equal to those made by a tangent with IF and IK.

Cor. The problem will be greatly simplified, if the given point A should occur in the straight line or in the circle, and hence coincide with either of the points of contact H or I; for EIH and FIK being drawn, the perpendicular HK is the diameter of the required circle.

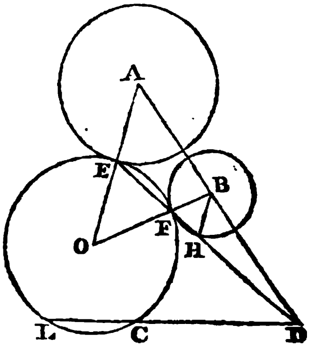
PROP. XXXV. PROB.

Through a given point, to describe a circle touching two given circles.

Let it be required, through the point C, to describe a circle touching two given circles whose centres are A and B.

ANALYSIS.

Join the centres A, B and O, and produce AB and the straight line connecting E, F, the points of contact, till they meet in D; join also BH and DC, and extend this to cut the circle in L.



Since EOF and FBH are isosceles triangles, the vertical angles OFE and BFH are equal to OEF and BHF, which are therefore equal, and consequently (I. 22. El.) EO is parallel to BH; whence (VI. 2. El.) AE : BH : : DA : DB, and the point D is therefore given. Again (VI. 1. El.) DA : DB : : DE : DH, or (V. 25. cor. 2. El.) DE.DF : DH.DF; but the rectangle DH, DF, being equal (III. 26. El.) to the rectangle under the segments of DB intercepted by the circle B, is given, and hence DE.DF or DC.DL are given rectangles; wherefore DC being given, DL and the point L are likewise given. The problem is thus reduced to Proposition 33. of this Book.

COMPOSITION.

Make (VI. 3. El.) AE : BH : : AD : BD, join DC, and produce it to L, such that the rectangle DC, DL shall be to the rectangle formed by a secant drawn from D to the circle B, in the ratio of AE to BH, and (II. 33. Anal.) describe a circle through the points C and L, and touching the circle A; this will also touch the circle B.

For join OA, OB, EH, and draw BH parallel to AO. Because AE : BH : : AD : BD, it is evident that EH, being produced, will meet AD in D; hence AE : BH : : DE : DH, or (V. 25. cor. 2. El.) DE.DF : DH.DF; but, by construction, AE : BH : : DC.DL : DH.DF, and consequently DC.DL is equal to DE.DF, and the point F lies in the circle O. Wherefore the triangle EOF is isosceles, and likewise the similar triangle HBF; hence F belongs also to the circle B and (III. 24. El.) is the point of mutual contact.

If L should coincide with the point C, the construction will be effected by the corollary to the preceding Proposition.

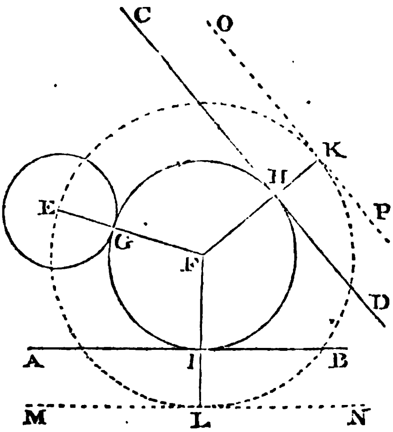
PROP. XXXVI. PROB.

To describe a circle that shall touch a given circle and two straight lines given in position.

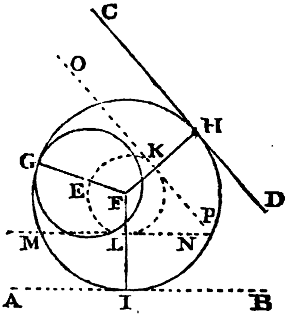
Let it be required to describe a circle touching the straight lines AB and CD, and another circle whose centre is E.

ANALYSIS.

Join FE, draw FH, FI to the points of contact, from F, with the radius FE, describe a circle meeting FH and FI produced in K and L, and, at these points, apply the tangents MN and OP.



Because FE = FK = FL and FG = FH = FI, therefore GE = HK = IL. But the tangents CD and OP, being perpendicular to FK, are parallel; and, for the same reason, the tangents AB and MN are parallel. Wherefore OP and MN are given in position, and consequently, by Prop 32. the circle EKL is given; and thence the concentric circle GHI.



COMPOSITION.

At a distance equal to the radius of the given circle, draw MN and OP parallel to AB and CD; and by Prop. 32. of this Book, find F the centre of a circle which passes through E and touches MN and OP; F is likewise the centre of the required circle.

For join FE, and draw FK and FL to the points of contact. And because GE=HK=IL, it is evident that FG =FH=FI. But the circle also touches at the points H and I, since CD and AB are perpendicular to FK and FL.

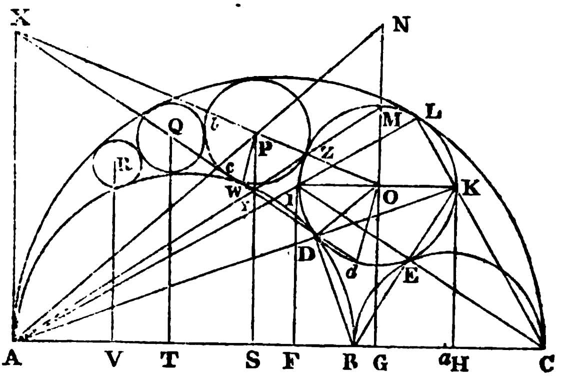
Scholium. The six preceding propositions are only cases of a general problem: " Three things being given,—whether points, or straight lines, or circles,—to describe a circle limited by them all." This problem comprises ten distinct cases. Two of these have been already given in the Elements: To describe a circle through three given points, forms the 9th Prop. Book III.: To describe a circle that shall touch three straight lines given in position, is the basis of Prop. 10. Book IV., and appears complete in the construction of Prop. 29. Book VI. The same principle, it may be perceived, runs through all the solutions already given; the conditions of the problem are only repeatedly exemplified, each of the linear or circular data being exchanged in succession for a point. Two cases still remain: When there are given three circles or two circles and a straight line, to describe another circle limited by these data. These are easily reduced, however, to the cases already solved, as in the concluding proposition,—by drawing a parallel, or describing a concentric circle, at distances, according to the relative position of the data, equal to the sum or difference of the given radii.

A beautiful general theorem respecting the mutual tangency of circles, was celebrated among the ancient geometers, and styled by them, from the resemblance of its diagram, the *αϱβηλος*, or *shoemaker's knife:* If, on the two segments AB and BC of the diameter of a semicircle, other semicircles be constructed, and between these and the exterior circumference a series of circles be inscribed, each of them touching in three several points, and if perpendiculars be let fall from their centres on the diameter AC; the first perpendicular OG will be equal to the diameter of the circle about O, the second perpendicular PS will be double of the diameter of the circle about P, the third perpendicular QT will be triple the diameter of the circle about Q, the fourth perpendicular RN will be quadruple the diameter of the circle about R, and so on continually. Inscribed circles might likewise be carried from O, on the other side towards C, or even downwards to B, whose diameter shall have the same continued relations to the perpendiculars from their centres.

The demonstration of this remarkable property, as it embraces a variety of circumstances, will necessarily extend to some length.

1. Join A, C, the extremities of the diameter of the ex-

terior circle, with L its point of contact with the first inscribed, and (III. 24. El.) the line IK, which connects their intersections with that circle must be parallel to AC; it is also a diameter, since ALC being contained in a semicircle, is (III. 19. El.) a right angle. For the same reason, if straight lines be drawn from B to the points of contact D and E, they must terminate in I and K, the extremities of the diameter IK. Now, from the nature of the circle (III. 26. El.) LA.AI =KA.AD; but the triangles LAC and FAI, having common angles at A, and the right angles ALC and AFI, are similar, and consequently AL : AC : : AF : AI, and (V. 6. El.) LA.AI=CA.AF. Moreover, the triangles HAK and DAB are evidently similar, and HA : AK : : AD : AB; whence HA.AB=KA.AD: Wherefore CA.AF=HA.AB, and AB : CA : : AF : HA, consequently (V. 11. El.) AB : BC : : AF : FH. Again (III. 26. El.) LC.CK =



IC.CE; but the triangles LCA and HCK being similar, AC : CL : : CK : CH, and AC.CH=LC.CK; and the triangles ICF and BCE being also similar, IC : CF : : BC : CE, and IC.CE = BC.CF; wherefore BC.CF = AC.CH, and AC : BC : : CF : CH; whence (V. 11. El.) AB : BC : : FH : CH. But it was proved that AB : BC t : : AF : FH, consequently AF : FH : : FH : CH, and $AF.CH=FH^2$. Again, the triangle ALC being evidently equiangular to AFI and KHC, these are mutually similar, and therefore AF : FI : : KH : CH, and $AF.CH=FI.KH$ or FI^2. Whence $FI^2=FH^2$, and the diameter IK equal to IF or to the perpendicular OG.

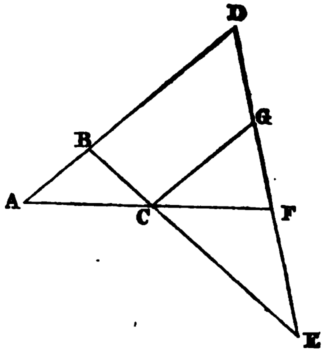
2. At A apply a tangent, and produce the line joining OP to meet it in X; join AP, and produce it to meet the extension of the perpendicular GO at N. Because AB : CA : : AF : HA, it follows (V. 12. El.) $AB+CA$: BC : : 2AG : FH, or having bisected BC in *a*, A*a* : B*a* : : AG : FG or OZ. In like manner, it may be shown, that A*a* : B*a* : : AS : PZ; whence by identity of ratios, AG : OZ : : AS : PZ, or, alternately, AG : AS : : OZ : PZ, and OZ : PZ : : AG : AS : : OX : PX; that is, the line OP joining two centres is cut internally in Z, and externally in X, in the same ratio. If PQ were joined, and produced to the tangent AX, it would likewise be divided in a similar way. But a line joining the points where the circles O and P touch the circle on AB, would, if produced, concur in the point X; for radii being drawn from the centres to the small intercepted chords would evidently form similar isosceles triangles, since these are equiangular to the isosceles triangle made by connecting the chord DW with the centre of the circle on AB. Again OD or OZ : P*c* or PZ : : OX : PX; and this analogy, by division, becomes OZ : PZ : : XZ : X*b*, and consequently (V. 3. El.) OZ : PZ : : XZ^2 : XZ.X*b*; but OZ : PZ : : OX : PX : : XD : X*c* : : XD.XW : X*c*.XW; wherefore, by identity of ratios, XZ^2 : XZ.X*b* : : XD.XW : X*c*.XW; but (III. 26. El.) $XZ.Xb=Xc.XW$, and hence $XZ^2=XD.XW=AX^2$, and $XZ=AX$. The triangle AXZ is therefore isosceles, and consequently (I. 22. El.) ZOM is also isosceles, and the line AZ meets the perpendicular

GO at M, the extremity of the diameter. And since $P\gamma : MN :: PA : NA :: PX : OX :: PZ : OZ :: P\gamma : OM$, therefore (V. 4. El.) MN is equal to OM, and $P\gamma$ to PZ. Now (VI. 2. El.) $P\gamma : PS :: NM : NG$ or $Zb : PS :: NO : NG$; but NG has been shewn to be double of NO, and consequently PS is double of Zb.

3. In the same way, it may be demonstrated that a line drawn from A through Q will meet the perpendicular SP at a distance above P, equal to the diameter of its circle; consequently the ratio of this perpendicular to the corresponding diameter being the same as that of the perpendicular QT to the diameter of the circle Q, is the ratio of three to one. The like reasoning will apply successively to the rest of the series of perpendiculars.

The process might easily be carried farther and more circles inserted, but it would render the figure very complicated. To avoid such confusion, likewise, the series of circles on the other side from O towards G along the concave circumference, and also a similar decreasing series of interior circles downwards from O to B, are omitted.

It may be worth while to notice here another elegant property connected with the tangency of circles: Let there be any three circles whose centres are A, B and C; if a pair of tangents, applied on both sides of the first and second circles, meet at the point D, and a pair of tangents applied to the second and third circles meet at E; then will the tangents applied to the first and third circles meet at some point F in the straight line DE. For let the diameters of the circles described about A, B and C, be expressed by a, b and c. It is evident, from the inspection of the figure in Case II. of the thirty-second Proposition of this Book, that each pair of tangents will meet in the extension of a straight line joining the centres of two circles, and divide externally the distance between these centres in the ratio of the diameters. Wherefore $a : b :: AD : DB$, and $b : c :: BE : EC$, but CG being drawn parallel to AD, $BE : EC :: BD : CG$, and by composition of ratios $a : c :: AD : CG :: $ (VI. 1. El.) $AF : FC$. Hence the tangents applied to the circle described from A and C meet also at F.



GEOMETRICAL ANALYSIS.

BOOK III.

DEFINITION.

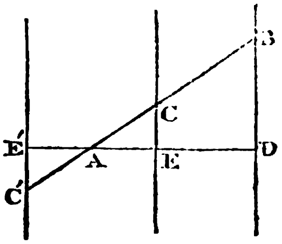
If a point vary its position according to some determined law, it will trace a line which is termed its *Locus.*

PROP. I. THEOR.

If a straight line, drawn through a given point to a straight line given in position, be divided in a given ratio, the *locus* of the point of section is a straight line given in position.

Let the point A and the straight line BD be given in position, and let AB, limited by these, be cut in a given ratio at C; this point will lie in a straight line which is given in position.

ANALYSIS.



From A let fall the perpendicular AD upon BD, and, through C, draw CE parallel to BD. It is evident (VI. 1. El.) that AC : AB : : AE : AD, and consequently that the ratio of AE to AD is given; but AD is given both in position and magnitude, and hence AE and the point E are given, and therefore CE, which stands at right angles to AD, is given in position.

COMPOSITION.

Let fall the perpendicular AD, which divide at E in the given ratio, and erect the perpendicular CE; this straight line is the *locus* required. For CE being parallel to BD, AC : AB : : AE : AD, that is, in the given ratio.

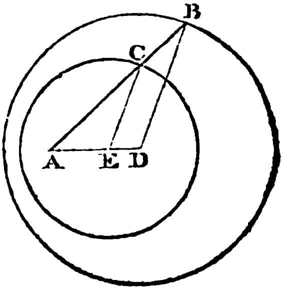
PROP. II. THEOR.

If a straight line, drawn through a given point to the circumference of a given circle, be divided in a given ratio, the *locus* of the point of section will also be the circumference of a given circle.

Let AB, terminating in a given circumference, be cut in a given ratio; the segment AC will likewise terminate in a given circumference.

ANALYSIS.

Join A with D the centre of the given circle, and draw CE parallel to BD. It is obvious (VI. 1. El.) that AC : AB : : AE : AD; whence the ratio of AE to AD being given, AE and the point E are given. Again, since (VI. 2. El.) AC : AB : : CE : BD, the ratio of CE to BD is given, and consequently CE is given in



magnitude. Wherefore the one extremity E being given, the other extremity of CE must trace the circumference of a given circle.

COMPOSITION.

Join AD, and divide it at E in the given ratio, and in the same ratio make DB to the radius EC, with which, and from the centre E, describe a circle.

For draw AB cutting both circumferences, and join CE and BD. Because CE : BD : : AE : AD, alternately CE : AE : : BD : AD; wherefore the triangles CAE and BAD, having likewise a common angle, are similar, and consequently AC : CB : : AE : AD, that is in the given ratio.

PROP. III. THEOR.

If, through a given point, two straight lines be drawn in a given ratio and containing a given angle; if the one terminate in a straight line given in position, the other will also terminate in a straight line given in position.

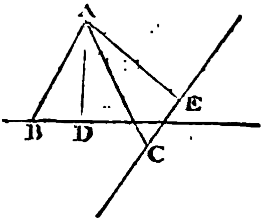
Let the ratio BA to AC, with the angle BAC and its vertex A, be given; if the extremity B lie in the straight line BD, the extremity C will have for its *locus* another straight line given likewise in position.

ANALYSIS.

Let fall the perpendicular AD upon BD, draw AE forming with AD an angle DAE equal to BAC, and make

AB : AC : : AD : AE; CE being joined, is the *locus* required.

Because the angle DAE is, by construction, equal to BAC, it is given; and the perpendicular AD being given, the straight line AE is, therefore, given in position. But AB : AC : : AD : AE, and this being a given ratio, AE is hence given also in magnitude. Again, since the angle BAC is equal to DAE, the angle BAD is equal to CAE; and because AB : AC : : AD : AE, alternately AB : AD : : AC : AE; wherefore the triangles ABD and ACE, having their vertical angles equal, and the sides containing those angles proportional, are (VI. 13. El.) similar, and consequently the angle CEA is equal to BDA, and therefore a right angle: consequently the straight line EC is given in position.



COMPOSITION.

Having let fall the perpendicular AD, and made the angle DAE equal to BAC, make AD to AE in the given ratio, and, at right angles to AE, draw EC; this is the *locus* required. For the triangles BAD and CAE, having their vertical angles equal, and the angles at D and E right angles, are similar, and consequently AB : AD : : AC : AE, or alternately AB : AC : : AD : AE, that is, in the given ratio.

PROP. IV. THEOR.

If, through a given point, two straight lines be drawn in a given ratio, and containing a given angle; if the one terminate in a given circumference, the other will also terminate in a given circumference.

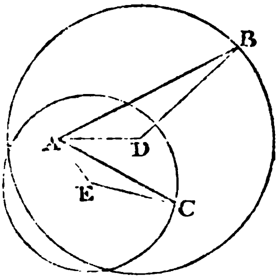
Let the angle BAC, its vertex A, and the ratio of its sides, be given; if AB be limited by a given circle, the *locus* of C will also be a given circle.

ANALYSIS.

Join A with D the centre of the given circle, draw AE at the given angle with AD, and in the given ratio, and join DB and EC.

Because the point A and the centre D are given, the straight line AD is given; and since the angle DAE, being equal to BAC, is given, AE is given in position. But AD being to AE in the given ratio, AE must be given also in magnitude, and consequently the point E is given.

Again, the whole angle BAC being equal to DAE, the part BAD is equal to CAE; and because AB : AC : : AD : AE; alternately AB : AD : : AC : AE; wherefore the triangles ADB and AEC are similar, and hence AB : BD : : AC : CE, or alternately AB : AC : : BD : CE; consequently the fourth term CE is given in magnitude; and its extremity E being given, the other must lie in a given circumference.



COMPOSITION.

Having drawn AE at the given angle with AD, make AD to AE in the given ratio, and in the same ratio let

DB be made to EC; a circle described from the centre E with the distance EC, is the *locus* required.

For AD : AE : : DB : EC, and alternately AD : DB : : AE : EC; but the angle BAD is equal to CAE, because the whole BAC is equal to DAE; consequently the triangles ABD and ACE are similar, and AB : AD : : AC : AE, or alternately AB : AC : : AD : AE, that is, in the given ratio.

Scholium. Since the tangent of a circle is only the extreme limit of its adjacent arc, which, in proportion as the circle expands, must continually approach to that ultimate position—the rectilineal, may be derived from the circular, *locus*. Thus, in Prop. 2. of this Book, if the centres E and D be supposed to retire to a distance indefinitely remote, the arcs which pass through C and B may be viewed as merging in their tangents or in perpendiculars let fall from those points upon AD, which is the first proposition. In like manner, if the circles in Prop. 4. be supposed immeasurably expanded, the arcs in which the points B and C lie may be conceived to pass into tangents perpendicular to AD and AE, as in Prop. 3.

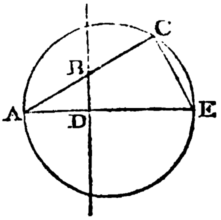
PROP. V. THEOR.

If a straight line, drawn from a given point to a straight line given in position, contain a given rectangle, the *locus* of its point of section will be a given circle.

Let the rectangle AB, AC be given, while the point B and the straight line BD are given in position; the point C will lie in the circumference of a given circle.

ANALYSIS.

Draw AD perpendicular to BD, and make the rectangle AD.AE=AB.AC. Since AD is evidently given both in position and magnitude, AE and the point E are given. Join CE. Because AD.AE = AB.AC, AD : AB : : AC : AE, and the triangles DAB and CAE, having the sides about the common angle at A proportional, are therefore similar; and consequently the angle ACE is equal to ADB, or a right angle. Whence (III. 19. El.) the point C must lie in a semicircle, of which AE the diameter is given.



COMPOSITION.

Having drawn the perpendicular AD, make the rectangle AD, AE equal to the given space, and upon the diameter AE describe a circle; this is the *locus* required. For draw AC and CE. The triangles ABD and AEC are similar, since they have a common angle at A, and those at D and C right angles; wherefore AB : AD : : AE : AC, and AB.AC=AD.AE, that is, equal to the given space.

PROP. VI. THEOR.

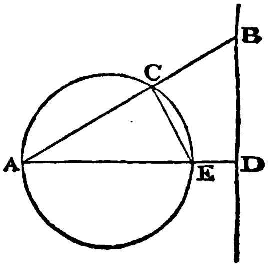
If a straight line, containing a given rectangle, be drawn through a given point to the circumference of a given circle, the *locus* of its point of section will be either a straight line given in position or a given circle, according as it orginates or not in the given circumference.

Let the rectangle AC, AB be equal to a given space, and the segment AC terminate in a given circumference, the point of origin A may lie either in that circumference or not.

1. *Suppose the given point A lies in the given circumference; the* locus *of B is a straight line given in position.*

ANALYSIS.

Draw the diameter AE, and make AE.AD=AB.AC; wherefore the point D is given. Join CE and BD; and because AE.AD=AB.AC, AC : AE : : AD : AB; whence the triangles CAE and DAB, having likewise a common angle at A, are similar. Consequently the angle ADB being thus equal to ACE, is a right angle, and the straight line DB is hence given in position.



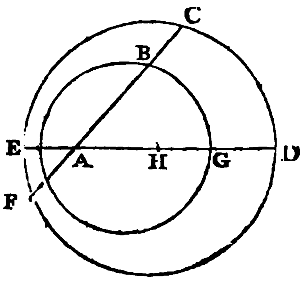
COMPOSITION.

Having drawn the diameter AE, make the rectangle AE, AD equal to the given space, and erect the perpendicular DB; this is the *locus* required. For draw ACB, and join CE. The right-angled triangles ACE and ADB being evidently similar, AC : AE : : AD : AB, and AC.AB= AE.AD, or the given space.

2. *Suppose that the point A does not lie in the given circumference; then the* locus *of B is a given circle.*

ANALYSIS.

Draw the diameter EAD, and produce CAF to the circumference. The rectangle AC, AF, being (III. 26. El.) equal to AD, AE, is given, and has, therefore, a given ratio to the rectangle AC, AB; whence (V. 3. El.) the ratio of AF to AB is given, and consequently (III. 2. Anal.) AB terminates in the circumference of a given circle.



COMPOSITION.

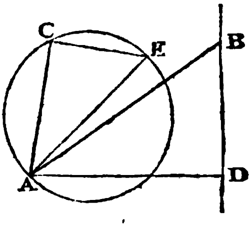
Having drawn the diameter EAD, make (II. 8. El.) the rectangle AD, AH equal to the given space, and (III. 2. Anal.) describe a circle EBGF, such that a straight line passing through it shall be cut by the circumference in the ratio of AE to AH; this circle is the *locus* required. For AE : AH : : AF : AB : : AF.AC : AB.AC; wherefore AF.AC : AB.AC : : AE.AD : AH.AD, and the first term of this analogy being equal to the third, the second term is equal to the fourth, or AB.AC=AH.AD, that is, equal to the given space.

PROP. VII. THEOR.

If two straight lines, containing a given rectangle, be drawn from a given point at a given angle; should the one terminate in a straight line given in position, the other will terminate in the circumference of a given circle.

Let the point A, the angle BAC, and the rectangle under its sides BA, AC be given; if the direction BD be given, then will the *locus* of C be a given circle.

ANALYSIS.



From A, let fall the perpendicular AD upon BD. Draw (I. 4. El.) AE, to contain with AD an angle equal to the given angle, and (II. 8. El.) a rectangle equivalent to the given space; and join CE.

Since AD is evidently given in position and magnitude, AE is likewise given in position and magnitude; and the rectangle AD, AE being equal to AB, AC, therefore (V. 6. El.) AD : AB : : AC : AE; but the angle DAE is, by construction, equal to BAC, and hence DAB is equal to EAC. Wherefore the triangles ABD and AEC, having each an equal angle and its containing sides proportional, are (VI. 13. El.) similar; and consequently the angle ACE is equal to the right angle ADB. Whence the *locus* of C is a circle, with AE for its diameter.

COMPOSITION.

Having let fall the perpendicular AD, draw AE, making the angle DAE equal to the given angle, and the rectangle DA, AE equivalent to the given space, and on AE, as a diameter, describe a circle; this is the *locus* required.

For join CE; and the triangles DAB and EAC being right-angled at D and C, and having the vertical angles at A equal, are evidently similar, and consequently AD : AB : : AC : AE; and hence the rectangle AB, AC is equal to AD, AE, that is, to the given space.

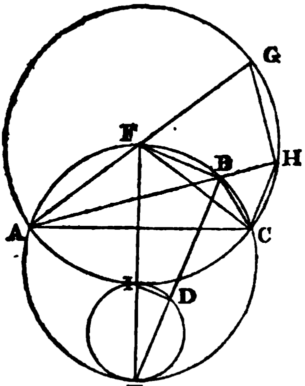
PROP. VIII. PROB.

To find the *locus* of the extremity of a straight line, which bisects the angle contained in a given segment of a circle, and is equal to half the sum of its sides.

Let BD, equal to half the sum of the sides AB and BC, bisect the angle ABC contained in the segment AFC; it is required to discover the *locus* of the point D.

ANALYSIS.

Having completed the arc AFC, bisect it in F, and with the chord AF or CF as a radius describe a circle cutting the diameter FE in I, produce AF and AB to G and H, and join GH, BF and DE.



Since the arc AF is equal to CF, the supplementary arc AE is equal to CE, and consequently BD, which bisects the angle ABC, forms one straight line with DE. Again, the angle ABC being equal (III. 16. El.) to AFC is (III. 15. El.) double of AHG or BHG; but (I. 30. El.) it is equal to both the angles BCH and BHC, and consequently these are mutually equal, and the triangle CBH is isosceles. Now the angle FAB or GAH is (III. 16. El.) equal to FEB, and the angles AHG and EBF being contained in semicircles, are right angles; wherefore the triangles AGH and EFB are similar, and AG : AH

: : EF : EB. But BD, being half the sum of AB and BC, is evidently the half of AH, and IF is equal to AF or the half of AG; consequently EF : EB : : IF : BD, and ID (VI. cor. 1. El.) must be parallel to FB; the angle IDE, being thus equal to FBE, is hence a right angle, and contained in a given semicircle. The *locus* of the point D is therefore given.

COMPOSITION.

Complete (III. 14. El.) the circle of which ABC is a segment; draw the diameter FE perpendicular to AC; make FI equal to the chord AF or EF, and on IE describe a circle; this circle will be the locus of D, the extremity of BD, which bisects the angle ABC, and is half the sum of the sides AB and CB.

For (III. 13. El.) the diameter EF evidently bisects the arcs AFC and AEC; so that BD being produced would terminate in E. About F, with the distance AF, IF or CF, describe a circle; produce AF and AB to G and H, and join GH, FB and ID.

The right-angled triangles AGH, EID and AFB, contained in semicircles, have, besides the angle GAH or FAB equal to FEB, and are consequently similar; wherefore AG : AH : : EF : EB : : (VI. 1. El.) EI : ED : : (V. 10. El.) : : FI : BD; but FI is the half of AG, and hence BD, which bisects the angle ABC, is the half of AH, that is, from what has been shown, the half of the sum of AB and BC.

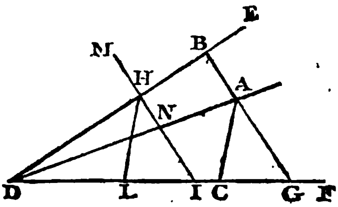
PROP. IX. THEOR.

If two straight lines, in a given ratio, stand at given angles on two diverging lines which are given in position, the *locus* of their vertex will be likewise a straight line given in position.

Let the straight lines AB, AC, in a given ratio, form given angles ABD and ACD with the given diverging lines DE, DF; then will their vertex A lie in a given direction.

ANALYSIS.

Join DA, and produce BA to meet DF in G. The triangle DBG is given in species; for the angles at D and B are given, and, consequently, the angle at G. Again, the triangle ACG is given in species, since all its angles are given. Hence the ratio of AC to AG is given; but the ratio of AB to AC is given, and consequently that of AB to AG and that of BG to AG are likewise given. Hence, also, the ratio of BG to DG is given, and therefore the ratio of AG to DG; and the angle at G being given, the triangle DAG is (VI. 13. El.) consequently given in species. Wherefore the angle GDA is given, and hence the straight line DA is given in position.



COMPOSITION.

In DE take any point H, and draw HI and HL, making with DE and DF angles equal to the respective inclinations of the bounded lines; produce IH to M, so that MH shall have to HL the given ratio; find (VI. 3. cor. El.) IN a third proportional to IM and IH, and join DNA; this straight line is the *locus* required.

Because IM : IH : : IH : IN, therefore (V. 10. El.) MH : IH : : NH : IN; but (VI. 2. El.) AB : AG : : NH : IN, and the triangles ACG and HLI being evidently similar, AG : AC :: IH : HL; therefore (V. 16. El.) AB : AC :: MH : HL, that is, in the given ratio.

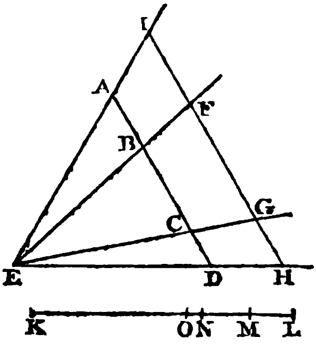
PROP. X. THEOR.

Three diverging lines being given in position, if a straight line cut them at given angles, and such that the rectangle of its first segment, by a given line, shall be equal to both the rectangles of its second and third segments by given lines; the *locus* of its point of origin will be a straight line given in position.

Let ABCD cut the diverging lines EF, EG and EH at given angles, and let AB.KL = AC.ML + AD.NM; then will the *locus* of the point A be a straight line given in position.

ANALYSIS.

Because AC.ML = AB.ML + BC.ML, and AD.NM = AB.NM + BD.NM, therefore AB.KL = AB.ML + BC.ML + AB.NM + BD.NM, and consequently AB.KL = AB (ML + NM) + BC.ML + BD.NM, and AB.KN = BC.ML + BD.NM. Make BC : BD :: NM : MO, and BC.MO = BD.NM; whence AB.KN = BC (ML + MO) = BC.OL, and AB : BC :: OL : KN. The ratio of AB to BC is therefore given; but the triangle BCE being given in species, the ratio of BE to BC is given, and consequently the ratio



of AB to BE is given; and since the contained angle ABE is given, the triangle BEA is likewise given in species; and thence the point A, and the straight line EA, are given in position.

COMPOSITION.

Having assumed in EH any point H, draw HGF in the given inclination, make $FG : FH :: NM : MO$, and produce HF till $KN : OL :: FG : IF$; EI is the straight line required. For $BC : AB :: FG : IF :: KN : OL$, and $AB.KN = BC.OL$; but $BC : BD :: FG : FH :: NM : MO$, and $BC.MO = BD.NM$. Wherefore $AB.KN = BC.OL = BC.ML + BD.NM$, and $AB.KM = AB.NM + BC.ML + BD.NM = BC.ML + AD.NM$, and hence $AB.KL = AB.ML + BC.ML + AD.NM = AC.ML + AD.NM$.

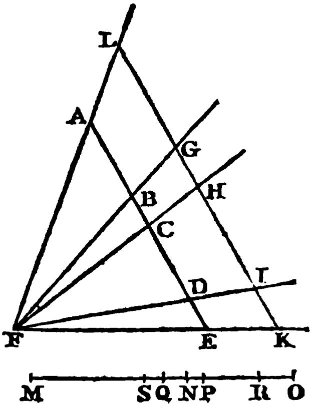
PROP. XI. THEOR.

Four diverging lines being given in position, if a straight line cut them at given angles, and such that the rectangles of its first and second segments by given lines shall be equal to both the rectangles of its third and fourth segments by given lines; the *locus* of its point of origin will be a straight line given in position.

Let $ABCDE$ cut the diverging lines FG, FH, FI, and FK at given angles, and let $AB.MN + AC.NO = AD.OP + AE.PQ$; then will the *locus* of the point A be a straight line given in position.

ANALYSIS.

Because $AB.MN + AC.NO = AD.OP + AE.PQ$, it follows, by decomposition, that $AB.MO+BC.NO=AB.OQ+BD.OP+BE.PQ$, and consequently $AB.MQ+BC.NO=BD.OP+BE.PQ$. Make $BD : BC :: NO : OR$, and $BD : BE :: PQ : PS$; then $BD.OR = BC.NO$, and $BD.PS = BE.PQ$; whence $AB.MQ+BD.OR = BD.OP + BD.PS$, or $AB.MQ = BD.SR$, and therefore, $AB : BD :: SR : MQ$. But the triangle BDF being given in species, the ratio of BD to BF is given; and consequently the ratio of AB to BF is given, and the contained angle ABF being given, the triangle BFA is likewise given in species; and hence the straight line FA is given in position.



COMPOSITION.

Having assumed in FK any point K, draw KIHG at the given inclination, make $GI : GH :: NO : OR$, and $GI : GK :: PQ : PS$, and produce KG till $MQ : SR :: GI : GL$; FL is the straight line required.

For $BD : BC :: GI : GH :: NO : OR$, and $BD.OR = BC.NO$; but $BD : BE :: GI : GK :: PQ : PS$, and $BD.PS = BE.PQ$; again, $MQ : SR :: GI : GL :: BD : AB$, and $AB.MQ = BD.SR$. Whence $AB.MQ+BC.NO = BD.SR+BD.OR = BD.SO = BD.PS+BD.OP = BE.PQ+BD.OP$, add to each $AB.OQ = AB.NQ+AB.NO$, or $AB.PQ+AB.OP$, and $AB.MN + AC.NO = AD.OP + AE.PQ$.

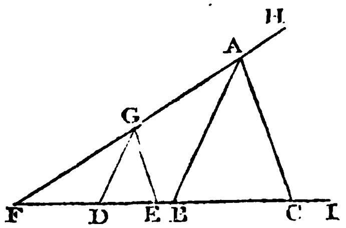
PROP. XII. THEOR.

If a straight line given in position, be cut at given angles by two straight lines, which intercept, from two given points in it, segments that have a given ratio, the *locus* of the point of concourse is a straight line given in position.

Let AB and AC be drawn, such that the angles ABF and ACF, with the ratio of DB to EC, are given; the *locus* of A, the point of concourse, is a straight line given in position.

ANALYSIS.

Make FD to FE in the given ratio, and join FA. Since therefore FD : FE : : DB : EC, it follows (V. 19. El.) that FD : FE : : FB : FC; consequently the ratio of FB to FC, and thence that of FB to BC, are each given. But the angles FBA and FCA being given, the triangle BAC is evidently given in species, and therefore the ratio of AB to BC is given, and hence the ratio of FB to AB is also given. The triangle FBA having thus two sides containing a given angle and in a given ratio, is (VI. 13. El.) given in species; and consequently the angle BFA is given, and the straight line FA given in position.



COMPOSITION.

Having made FD to FE in the given ratio, draw DG and EG at the given angles with FI, and join F with their point of concourse; FGH is the *locus* required.

For, from any point A in FH, draw AB and AC at the given angles with FI, and consequently parallel to GD and GE. Because AB is parallel to GD, and AC to GE, FG : FA : : FD : FB : : FE : FC (VI. 1. El.), and alternately FD : FE : : FB : FC; wherefore (V. 19. cor. 1. El.) DB : EC : : FD : FE, that is, in the given ratio.

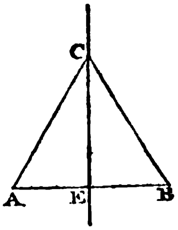
PROP. XIII. THEOR.

If, from two given points, there be inflected two straight lines in a given ratio, the *locus* of their point of concourse is a straight line, or a circle given in position.

Let AC and BC, drawn from the points A and B, have a given ratio; then will C, the point of concourse, lie in a straight line given in position, or in the circumference of a given circle.

1. *When the inflected lines are equal, they terminate in a straight line given in position.*

ANALYSIS.



Bisect AB in E, and join EC. The triangles ACE and BCE, having the sides AE and AC equal to BE and BC, and EC common, are equal (I. 2. El.); wherefore the angle AEC is equal to BEC, and EC is perpendicular to AB, and consequently given in position.

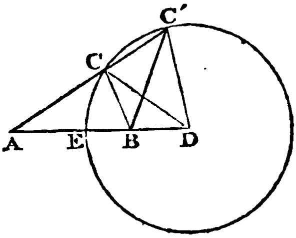
COMPOSITION.

Bisect AB by the perpendicular EC, which is the *locus* required. For draw AC and BC to any point in it, and the triangles AEC and BEC are (I. 3. El.) evidently equal, and hence AC is equal to BC.

2. *When the inflected lines AC and BC have a ratio of inequality, their point of concourse lies in the circumference of a given circle.*

ANALYSIS.

Draw CD, making the angle BCD equal to BAC, and meeting AB produced in D. The triangles DAC and DCB, having the angle at D common, and the angles at A and C equal, are evidently similar; and hence AD : AC : : DC : BC, and alternately AD : DC : : AC : BC, that is, in the given ratio. But AD : DC : : DC : BD, and consequently AD is to BD in the duplicate of the given ratio of AD to DC, and which is therefore likewise given. Consequently BD, and the point D, are given; and DC being thence given, its extremity C must lie in the circumference of a circle described with that radius.



COMPOSITION.

Divide AB (VI. 4. El.) in the given ratio at E, and in the same ratio make ED to BD; the circle described from the centre D and with the radius DE, is the *locus* required.

For, since AE : BE : : ED : BD, it follows (V. 19. El.) that AD : ED or DC : : ED or DC : BD; hence the triangles DAC and DCB, having the sides which contain their common angle at D proportional, are similar, and therefore AC : AD : : BC : DC, or alternately AC : BC : : AD : DC or ED, that is, in the given ratio.

Scholium. Since, in the second case, AC : BC : : AD : ED, it is obvious, that as the ratio of AC to BC approaches to equality, the centre D must continually recede from A or E, and consequently the arc EC may be conceived as ultimately passing into the tangent which bisects AB at right angles; thus comprehending the first case of the proposition.

PROP. XIV. PROB.

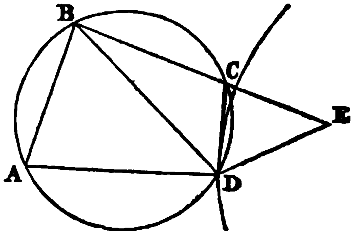
To inscribe in a circle a quadrilateral figure, of which the sides are given.

Let it be required to find a circle that will circumscribe the quadrilateral figure ABCD, contained by given sides.

ANALYSIS.

Join the diagonal BD, from D draw (I. 4. El.) DE making with the side CD an angle CDE equal to ABD, and meeting BC produced in E.

The exterior angle DCE of the inscribed quadrilateral (III. 17. cor. 2. El.) is equal to BAD, and the angle DCE is by construction equal to ABD; wherefore the triangles ADB and DEC (VI. 11. El.) are similar, and AB : AD :: CD : CE; but the three terms of this analogy being given, the fourth (VI. 3. El.) is also given, and thence the point E. Again, from the similitude of the same triangles, AB : BD :: CD : DE, and alternately AB : CD :: BD : DE; but AB and CD being both given, their ratio is given, and consequently that of BD to ED. The points B and E being thus given, and likewise the ratio of the inflected lines BD and ED, the *locus* of their concourse D is, by the last



proposition, a given circle. But CD is given, and therefore the point D, and the three points B, C, and D, being all given, the circle which passes through them (III. 9. El.) is hence given.

COMPOSITION.

Let L, M, N, and O express in succession the several sides of the quadrilateral figure; make L : M : : N : CE, and annex this in the same straight line with the fourth side O or BC; describe, by the preceding proposition, a circle, which is the *locus* of lines inflected from B and E in the ratio of L to N; from C inflect to that circle CD equal to N; about the three points B, C, and D, describe another circle, in which, from B, insert AB equal to L; and AD, being joined, will be equal to M. For join BD and ED. Since, from the property of *loci*, AB : CD : : BD : DE, alternately AB : BD : : CD : DE, and (III. 17. cor. 2. El.) the angle ECD being equal to BAD, the triangles ABD and CDE are similar, and AB : AD : : CD : CE; but by construction AB : M : : CD : CE, and consequently (V. 17. El.) AD : M : : CD : CD, and AD is equal to M. The circle described admits, therefore, all the four chords which form the sides of the quadrilateral figure.

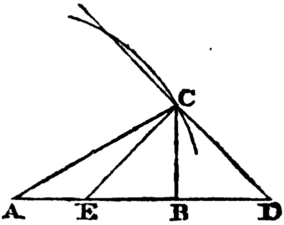
PROP. XV. PROB.

Given the hypotenuse of a right-angled triangle, and the sum or difference of the base and perpendicular, to construct the triangle.

ANALYSIS.

In the base AB, or its production, make BD or BE equal to the perpendicular BC, and join CD or CE.

The triangles CBD and CBE are right-angled and isosceles, and therefore the angles at D and E are each of them half a right angle. If AD, the sum of AB and BC, be given, the point D is given, and consequently the straight line DC, making a given angle with DA, is given in position; or if AE, the difference between the base and perpendicular, be given, the point E is given, and the straight line EC is given in position. But the hypotenuse AC being given, the point C must, therefore, occur in the contact or intersection of a circle described from A with that radius and the straight line CD or CE. Consequently C is given, the perpendicular CB, and thence the right-angled triangle ABC.



COMPOSITION.

Make AD or AE equal to the sum or difference of AB and BC; draw (I. 5. and 4. El.) DC or EC at an angle CDE or CED equal to half a right angle; from A with the radius AC describe a circle meeting DC or EC in the point C, and from C (I. 6. El.) let fall the perpendicular CB: ACB is the triangle required.

For the right angled triangles CBD and CBE are evidently isosceles, and therefore AD is equal to the sum, and AE to the difference, of AB and BC.

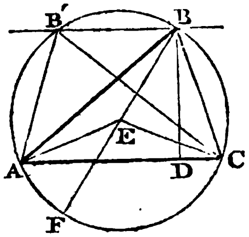
PROP. XVI. PROB.

Given the base of a triangle, its altitude, and the rectangle under its two sides,—to determine the triangle.

ANALYSIS.

About the triangle ABC describe (III. 9. cor. El.) a circle, and draw the diameter BF and the radii AE and CE.

Because the given rectangle AB.BC is (VI. 18. El.) equivalent to BD.BF, this rectangle is likewise given; and since the perpendicular BD is given, the diameter BF, and therefore the radii AE, CE, are given. But the base AC being given, the triangle AEC is hence given, and consequently the centre E and the circle ABCF are given. Again, because BD, the distance of the vertex of the triangle from its base, is given, that point must occur in the parallel BB′, and, being thus placed in the contact or intersection of a given straight line with a given circle, is itself given.



COMPOSITION.

On BD construct (II. 9. El.) a rectangle equivalent to the given space; also form on AC the triangle AEC, having AE and CE each equal to half the greater side of that rectangle; from E with the radius EA describe a circle; on AC erect a perpendicular DB equal to the altitude of the triangle, and through B draw a parallel meeting the circumference in B or B′: ABC is the triangle required.

For ABC has evidently the given altitude BD, and the rectangle AB.BC, being equivalent (VI. 18. El.) to BF.BD, is therefore equivalent to the given space.

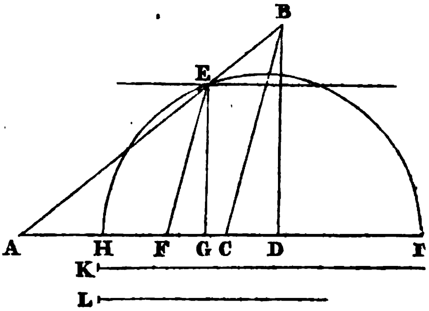
PROP. XVII. PROB.

To construct a triangle which has two sides given, and its base equal to the altitude.

Let the triangle ABC have its base AC equal to the altitude BD, and let the sides AB and BC be equal to the given lines K and L, to construct the triangle.

ANALYSIS.

Assume any base AF; draw FE parallel to CB meeting AB or its extension, and let fall the perpendicular EG. It is evident (VI. 1. El.) that AC : AF : : AB : AE : : BD : EG; but BD is by hypothesis equal to AC, and therefore (V. 4. El.) EG is equal to AF, and is consequently given in magnitude. Wherefore the point E lies in a parallel to AC at the distance AF. Again, the ratio of AE to EF, being the same as that of AB to BC, is given, and consequently (by Prop. 13. of this Book,) the locus of E is a given circle. The triangle AEF is hence given; but the side AB being given, the similar triangle ABC is also given.



COMPOSITION.

Having assumed a base AF, divide (VI. 4. El.) it internally and externally at H and I in the ratio of K to L; on HI describe a semicircle, and at a distance equal to AF draw a parallel meeting the circumference in E; join AE and FE; make AB equal to K, and draw BC parallel to EF; ABC is the triangle required.

For K : L : : AE : FE : : AB : BC; but AB was made equal to K, and therefore BC is equal to L. Again, AF : AC : : AE : AB : : EG : BD, and since EG is by construction equal to AF, the altitude BD must be equal to the base AC.

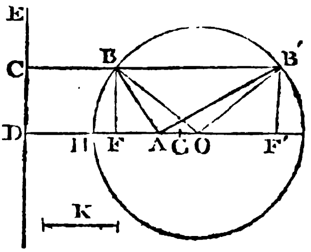
PROP. XVIII. THEOR.

A point and a straight line being given in position, the *locus* of another point, the square of whose distance from the former is equal to the rectangle under its distance from the latter and a given straight line—is a given circle.

The point A and the straight line DC being given in position, let the square of BA be equal to the rectangle under the perpendicular BC and K; the *locus* of B is a given circle.

ANALYSIS.

Draw DFA parallel to CB, make AO equal to the half of K, and bisect it in G, join BO, and let fall the perpendicular BF.



Because AO is bisected in G, $OB^2 - AB^2$, or $AB'^2 - OB'^2$, (II. 21. El.) $= 2AO.GF = K.GF$; but $AB^2 = K.BC$, or $K.DF$, and hence $OB^2 = K.DG$. Since therefore DG is given, OB is also given; and the one extremity O being given, the other extremity B must lie in the circumference of a given circle.

COMPOSITION.

Having drawn DA parallel to CB, make $AO=\frac{1}{2}K$, and $AG=\frac{1}{2}AO$, and find OH a mean proportional between K and DG; a circle described from O with the radius OH is the *locus* required.

For OB^2-AB^2, or $AB'^2-OB'^2$, $=2AO.GF=K.GF$; and since, by construction, OH^2 or $OB^2=K.DG$, it follows that $AB^2=K.DF$ or K.BC.

Cor. If the given point A lies in DC, or coincides with D, then $DG=\frac{1}{4}K$ and $OH=\frac{1}{2}K$, or the circle likewise passes through D; whence AB becomes a chord, and its square (VI. 15. cor. 1. El.) is equivalent to the rectangle under the segment DF, and the diameter or K.

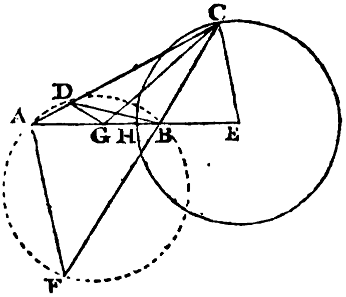
PROP. XIX. THEOR.

If, from two given points, there be inflected two straight lines, such that the difference of the square of the one and a given space shall have to the square of the other, a given unequal ratio—their point of concourse will lie in the circumference of a given circle.

Let AC and BC be the inflected lines, and the rectangle AC, AD be made equal to the given space; then if the difference between the square of AC and that rectangle, or the remaining rectangle AC, CD, have a given unequal ratio to the square of BC, the *locus* of the point C will be a given circle.

ANALYSIS.

Make (VI. 4. El.) AE to BE in the given ratio; join CE and BD; produce CB to meet the circumference of a circle described about the triangle ADB, and join AF.



Because (III. 26. El.) the rectangle AC, CD is equal to FC BC, it follows that the rectangle FC, BC is to the square of BC, or (V. 25. cor. 2. El.) FC is to BC, in the given ratio of AE to BE; wherefore (VI. 1. cor. 1. El.) AF is parallel to CE, and consequently the angle ECB is equal to AFB, which is equal to CDB, the opposite exterior angle of the quadrilateral figure ADBF. Through the points C, D, B, describe a circle cutting AB in G, and join CG and DG; then (III. 26. El.) the rectangle BA, AG is equal to CA.AD, or to the given space, and hence AG, and the point G are given. The angle CDB, or ECB, is, therefore, equal to CGB, and consequently the triangles BEC and CEG are similar, and GE : CE : : CE : BE; whence $CE^2 = GE.BE$, which is a given rectangle, and thus CE is given, and consequently the *locus* of C is a given circle.

COMPOSITION.

Make (III. 8. El.) the rectangle AB, AG equal to the given space, and (VI. 4. El.) AE to BE in the given ratio, and find (VI. 16. El.) EH a mean proportional between GE and BE; the *locus* required is a circle described from E with the radius EH.

For, (III. 9. cor. El.) through the points A, D, B, and through C, B, G, describe circles; produce CB to F, and join AF, CG, and DG. Because $GE.BE=HE^2$, therefore (V. 6. El.) GE : HE or CE : : HE or CE : BE, and hence the triangles GEC and CEB, having a common vertical angle, are (VI. 13. El.) similar, and the angle EGC is equal to ECB; but the angle EGC, or BGC, is equal to CDB, which again is equal to AFB; consequently the alternate angles ECB and AFB are equal, and (I. 22. El.) the straight lines CE and AF parallel. Wherefore AE : BE : : FC : BC : : FC.BC or AC.CD : BC^2. But CA.AD$=$BA.AG, or the given space; and hence the difference between the square of AC and that space, or the rectangle AC, CD, is to the square of BC, in the given ratio.

Scholium. If this local theorem were extended to the extreme cases, it would include other propositions which are generally exhibited in a separate form. Thus, supposing the given ratio to be that of equality, the sum or difference of the squares of AC and BC will be equivalent to the given space, according as this is greater or less than the square of AC. When the given space exceeds the square of AC, the centre E of the circle bisects AB, as in the first case of the twenty-second proposition of this Book. But when the square of AC is deficient by the given space, the ratio of AE to BE being that of equality, the centre E, lying beyond B, must be thrown to an infinite distance, and consequently the arc which crosses AB will merge in a tangent bisecting GB at right angles, as in Proposition 20. Again, if the deficient space be supposed to vanish, while the ratio of the squares of AC and BC, or that of the inflected lines themselves is given, the point G will coincide with A, and the centre and radius of the circle are hence determined, after the same manner as in Proposition 13.

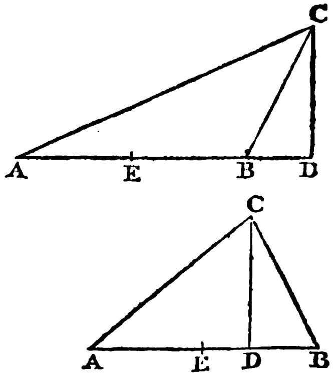
PROP. XX. THEOR.

If from two given points there be inflected two straight lines, of whose squares the difference is given, the *locus* of their point of concourse will be a straight line given in position.

Let AC and BC, drawn from the points A and B, have the difference of their squares given; the *locus* of C, the point of concourse, is a straight line given in position.

ANALYSIS.

Draw CD perpendicular to AB, which bisect in E. The difference between the squares of AC and BC is (II. 21. El.) equal to twice the rectangle under AB and ED; consequently that rectangle, and its containing side ED, are given; whence the point of bisection E being given, the point D is given, and the perpendicular CD is therefore given in position.



COMPOSITION.

Bisect AB in E, and make (II. 8. El.) the rectangle under twice AB and ED equal to the given space; the perpendicular DC is the *locus* required.

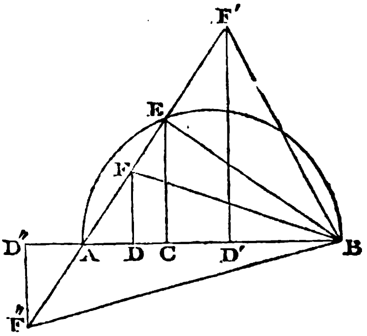
For (II. 21. El.) $AC^2 - BC^2 = AB.2ED = 2AB.ED$, and consequently the difference of the squares of AC and BC is equal to the given space.

LEMMA.

*If a straight line AB be cut any how in the point D, but divided at C, so that the segment AC shall be the n*th *part of BC; then* $n.AD^2+BD^2=AB.BC+(n+1)CD^2$.

For upon AB describe a semicircle, and erect the perpendicular CE; join AE, BE; draw DF parallel to CE and meeting AE or its extension, and join BF.

The angle AEB in a semicircle being a right angle, AC : CE : : CE : BC (VI. 15. cor. El.) and consequently (V. 24. El.) AC : BC : : $AC^2 : CE^2$; but $BC=n.AC$, and therefore $CE^2=n.AC^2$. Again, from the same property, AB : AE : : AE : AC, and AB : AC : : $AE^2 : AC^2$; and since $AB=(n+1)AC$; it follows (V. 5. El.) that $AE^2=(n+1)\ AC^2$. Now CE and DF being parallel, CE : DF : : AC : AD, and (V. 22. cor. 1. El.) $CE^2 : DF^2 : : AC^2 : AD^2$, and CE^2 being equal to $n.AC^2$, therefore (V. 8. and 5. El.) $DF^2=n.AD^2$. In the same manner, it is shown that $EF^2=(n+1)\ CD^2$. But (VI. 15. cor. 1. El.) $BE^2=AB.BC$, and the triangles BDF and BEF being right angled, $BD^2+DF^2=BF^2=BE^2+EF^2$, and consequently by substitution, $n.AD^2+BD^2=AB.BC+(n+1)CD^2$.



PROP. XXI. THEOR.

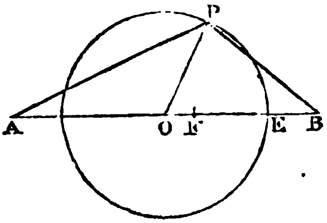
If, from given points, there be inflected straight lines, whose squares are together equal to a given space,—their point of concourse will terminate in the circumference of a given circle.

1. *When there are only two given points.*

Let AP and BP, drawn from the points A and B, have the sum of their squares given; the *locus* of their point of concourse is a given circle.

ANALYSIS.

Bisect AB in O, and join OP. The squares of AP and BP are (II. 22. El.) equal to twice the squares of AO and OP. Hence the sum of the squares of AO and OP is given; but AO and its square being given, the square of OP and OP itself must be given; wherefore the *locus* of the extremity P is a circle, of which the point of bisection is the centre.



COMPOSITION.

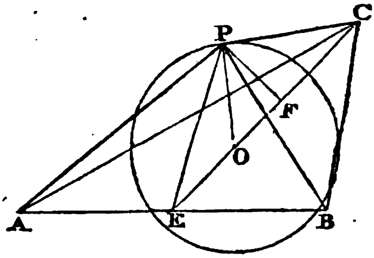
Bisect AB in O; find (III. 27. El.) AF the side of a square equal to half the given space, and make (II. 13. El.) $OE^2 = AF^2 - AO^2$; the point O is the centre, and OE the radius, of the required circle.

For (II. 22. El.) $AP^2 + BP^2 = 2AO^2 + 2OP^2 = 2AO^2 + 2OE^2 = 2AF^2$, or the given space.

2. *When three points are given.*

Let the straight lines AP, BP and CP, inflected from the points A, B, and C, have the sum of their squares given; the *locus* of their point of concourse is a given circle.

ANALYSIS.



Bisect AB in E, and (II. 22. El.) $AP^2+BP^2=2AE^2+2EP^2$; consequently $AP^2+BP^2+CP^2=2AE^2+2EP^2+CP^2$. Now $2AE^2=AB.BE$, and, letting fall the perpendicular PF, (II. 10. El.) $2EP^2=2EF^2+2PF^2$, and $CP^2=PF^2+CF^2$. Wherefore $AP^2+BP^2+CP^2=AB.BE+3PF^2+2EF^2+CF^2$. Trisect EC (I. 36. El) in the point O, and join PO; and, by the *Lemma*, $2EF^2+CF^2=EC.CO+3OF^2$. Whence $AP^2+BP^2+CP^2=AB.BE+EC.CO+3PF^2+3OF^2=AB.BE+EC.CO+3PO^2$. But the intermediate points of division E and O are evidently given, and thence the rectangles AB, BE and EC, CO, are given; wherefore $3PO^2$ is given, and consequently PO itself. Since one extremity of that line then is given, the other extremity P must lie in the circumference of a given circle.

COMPOSITION.

Bisect AB in E, trisect EC in O, and find (III. 27. El.) OP such that its square shall be triple the excess of the given space above the rectangles AB, BE and EC, CO; the *locus* required is a circle, of which O is the centre, and PO the radius. For $3PO^2=3PF^2+3OF^2$, $3PO^2+$

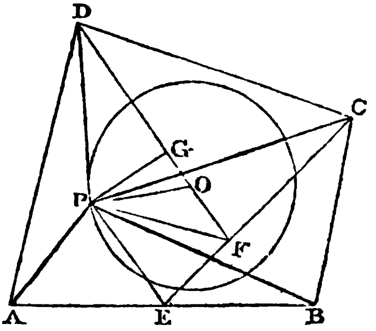
$EC.CO = 3PF^2 + EC.CO + 3OF^2 = 3PF^2 + 2EF^2 + CF^2 = 2PE^2 + PF^2 + CF^2 = 2PE^2 + CP^2$; consequently the given space, or $3PO^2 + AB.BE + EC.CO = 2AE^2 + 2PE^2 + CP^2 = AP^2 + BP^2 + CP^2$.

3. *When there are four given points.*

Let AP, BP, CP and DP drawn from the points A, B, C, and D, have the sum of their squares given; the *locus* of their concourse P is a given circle.

ANALYSIS.

Bisect AB in E, trisect EC in F, and join PE and PF. It is manifest, from the last case, that $AP^2 + BP^2 + CP^2 = AB.BE + EC.CF + 3PF^2$; add DP^2 to each, and $AP^2 + BP^2 + CP^2 + DP^2 = AB.BE + EC.CF + 3PF^2 + DP^2$. Let fall the perpendicular PG upon DF, and the given space is equal to $AB.BE + EC.CF + 3PG^2 + 3FG^2 + PG^2 + DG^2$; and hence $4PG^2 + 3FG^2 + DG^2$ must be equal to a given space. Let FO be made the fourth part of DF, and join PO: then, by the *Lemma*, $3FG^2 + DG^2 = FD.DO + 4OG^2$. Wherefore $FD.DO + 4OG^2 + 4PG^2$, or $FD.DO + 4PO^2$, is equal to a given space, and hence $4PO^2$, and PO itself are given. Now the point O being given, P must lie in the circumference of a given circle.



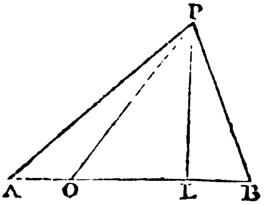
COMPOSITION.

Bisect AB in E, trisect EC in F, and quadrisect FD in O; from the given space take away the accumulate rectangles $AB.BE+EC.CF+FD.DO$, and find (III. 27. El.) the side of a square equal to this difference: That straight line is the diameter of a circle, which is the *locus* required.

For join PE, PF, PO, and let fall the perpendicular PG upon DF; then $FD.DO+4PO^2=FD.DO+4OG^2+4PG^2=3FG^2+DG^2+4PG^2=3FG^2+3PG^2+DP^2=3PF^2+DP^2$. Wherefore $AB.BE+EC.CF+3PF^2+DP^2$, is equal to the given space. But, from the composition of the last case, it is manifest that $AP^2+BP^2+CP^2=AB.BE+EC.CF+3PF^2$; consequently $AP^2+BP^2+CP^2+DP^2$ are together equal to the given space.

By pursuing this mode of investigation, it is obvious that the proposition will be successively extended to any number of given points.

Scholium. The property now demonstrated is capable of being generalized. Thus, if any multiples of the squares of the inflected lines be together equal to a given space, the *locus* of their point of concourse is still a given circle: For, conceive so many points to be clustered together at each centre A, B, C, &c. of inflection, and the squares of the lines which proceed from them will evidently receive in effect a corresponding multiplication.—But the property may be traced out more clearly, and through all its shadings, by help of a simple extension of the *Lemma*. Let AP and BP be two straight lines inflected from the points A and B, and let the segment $OB=v.OA$; then, joining PO



and drawing the perpendicular PL, it was proved that $v.AL^2+BL^2=AB.BO+(v+1)OL^2$; add $(v+1)\ PL^2$ to each, and $v(AL^2+PL^2)+BL^2+PL^2=AB.BO+(v+1)(OL^2+PL^2)$, or $v.AP^2+BP^2=AB.BO+(v+1)OP^2$. Multiply both by n, and suppose $nv=m$, and there results $m.AP^2+n.BP^2=n.AB.BO+(m+n)\ OP^2$. By repeated application of this principle, it may be demonstrated that $m.AP^2+n.BP^2+p.CP^2+q.DP^2$, &c. $=(m+n+p+q,$ &c.$)\ OP^2$, together with certain multiples of given rectangles, and consequently that their point of concourse has for its *locus* a circle, whose centre is O and radius OP. But the property must likewise hold, if all those multiple squares were divided by the same number, that is, if instead of the squares of the inflected lines, there were substituted only similar rectilineal figures constructed upon them. If the given space should be equal to the rectangles, the circle will evidently contract to a point, and beyond this limit the problem becomes impossible. It is likewise obvious, that the centre O and radius OP will turn out the same, in whatever order the successive connected sections take place.

DEFINITION.

A *Porism* proposes to demonstrate that one or more things may be found, between which and innumerable other objects assumed after some given law, a certain specified relation is to be shown to exist.

The nature of a porism consists in affirming the possibility of finding such conditions, as will render a problem indeterminate, or capable of innumerable solutions.

PROP. XXII. PROB.

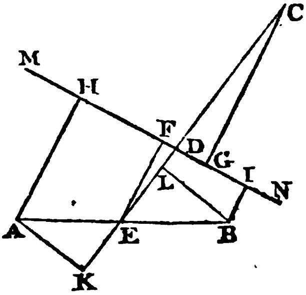
Three points being given, a fourth may be found, such that any straight line drawn through it shall have its distances from two of those equal to its distance from the third.

Let A, B, and C, be given points, another point D may be found, so that HDI being drawn through it, the perpendiculars AH and BI, let fall on the one side, shall be equal to the perpendicular CG on the other side.

ANALYSIS.

Through the point D, draw CDK, and upon this let fall the perpendiculars AK, BL, and join AB, meeting KC in E.

Since CDK passes through C, its distances KA and LB, on either side, from the two remaining points, must evident-



ly, from the hypothesis, be equal. Hence (I. 20. El.) the right-angled triangles AEK and BEL are equal, and consequently the side AE is equal to BE; wherefore E, being thus the point of bisection, is given. Draw the perpendicular EF; and it is evident (II. 9. El.) that 2EF= AH and BI. Now CG and EF being parallel, CD : DE : : CG : EF, and (V. 13. El.) CD : 2DE : : CG : 2FE or AH+BI; but, by hypothesis, CG=AH+BI, and therefore (V. 4. El.) CD=2DE. Whence, CE being given, the point D is given.

COMPOSITION.

Bisect AB in E, join CE and trisect it in D; this is the point required.

For let fall the perpendicular EF. Because CG and EF are parallel, CD : DE : : CG : EF; but CD=2DE, and therefore (V. 4. El.) CG=2EF, that is, AH+BI.

The porism now demonstrated may be viewed as originating in the solution of this problem :—To draw, through the point M, a straight line MN, such that the perpendiculars AH and BI, let fall upon it from the points A and B, shall be together equal to the perpendicular CG, from the point C on the other side. The point D is found as before, and thence the position of MDN is assigned. But this straight line, it is evident, will become indeterminate if the point M should happen to coincide with D; on that supposition, the problem would admit of innumerable answers, or the diameter MDN might lie in every possible direction.

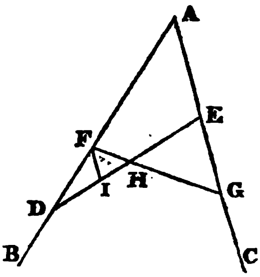
PROP. XXIII. PORISM.

A straight line contained within a given angle may be found, such that any other containing line which is cut by it in a given ratio shall intercept from the sides of the angle segments that are together equal to a given straight line.

Let BAC be a given angle, a straight line DE contained within it may be found, such that any other line FG cut by it in the given ratio of FH to HG, shall intercept segments AF and AG, whose sum is equal to a given line.

ANALYSIS.

Since the intercepting line FG is indeterminate, DHE may be viewed as one of its positions, and consequently the sum of AD and AE is equal to the given line, or to the sum of AF and AG; wherefore the excess DF on the one side is equal to the defect EG on the other. Draw FI parallel to AC, and (VI. 2. El.) AE : AD : : FI : DF or EG; but FH : HG : : FI : EG or DF, that is, AE : AD; but the ratio of FH to HG is given, and therefore that of AE to AD, whose sum is also given; whence these segments themselves are (VI. 4. El.) each given. The points D, E are consequently given, and the straight line DE.



COMPOSITION.

Divide the line equal to the sum of the segments in the given ratio, and make AD and AE equal to the two portions; DE, being joined, will be the line required. For, assume any point F in AD, cut off GE=DF, and join FHG: This line, which obviously forms segments AF and AG equal to the whole sum, is likewise divided at H in the given ratio of AE to AD. For draw FI parallel to AC, and FH : HG : : FI : EG or DF : : AE : AD. The ratio of the segments FH to HG, being thus the same as that of AE to AD, is hence given.

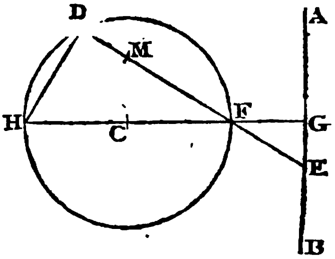
PROP. XXIV. PORISM.

A circle and a straight line being given in position a point may be found, such that any straight line, drawn through it and limited by these, shall contain a given rectangle.

Let the straight line AB, and the circle HDF, be given in position; it is required to determine a point F, which may divide any connecting straight line DFE into segments containing a rectangle that will be given.

ANALYSIS.

Through F draw HFG perpendicular to AB. By hypothesis, the rectangle HF.FG is likewise equal to the given space, and therefore equal to DF.FE; whence (V. 6. El.) DF : HF : : FG : FE, and the triangles DFH and



GFE, having the vertical angles at F equal, are (VI. 13. El.) consequently similar, and the angle FDH is thus equal to FGE, or is a right angle. Wherefore HDF is a semicircle, of which HF is the diameter; but the centre C being given, the perpendicular HCG is thence given, and consequently the extremity of the diameter, or the point F. Again, the points H, F, and G being given, the rectangle under the segments HF and FG is given.

COMPOSITION.

From the centre C, let fall upon AB the perpendicular HCFG, cutting the circumference in F; this point has the property, that any intersecting line drawn through it will contain a given rectangle. For join DH, and the triangles FGE and FDH are similar; whence FG : FE : : FD : FH, and consequently FE.FD=FG.FH, which is manifestly given.

This porism also may be considered as arising out of the solution of a simple problem:—Through the point M, to draw a straight line DMFE, so that its segments DF and FE, intercepted between a circle and a straight line, shall contain a given rectangle. The point F being found as before, DME is consequently given in position. But when the point M coalesces with F, the straight line DE can thus have no determinate position, or it will fulfil the conditions of the problem in whatever direction it be drawn.

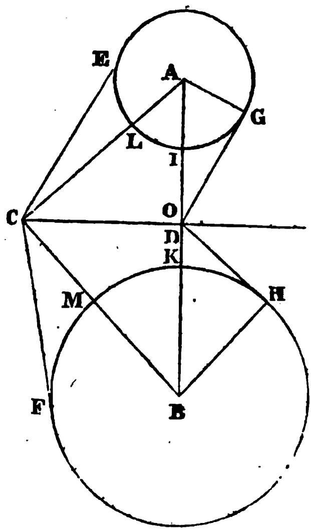
PROP. XXV. PORISM.

A straight line may be found, from any point of which two tangents drawn to given circles that do not meet shall be equal.

Let A and B be the centres of two given circles, a straight line CD may be found, such that the tangents CE, and CF drawn from any point C shall be equal.

ANALYSIS.

Let CD cut the straight line joining the centres in D; and this point must have the same property as C, or the tangent DG will be equal to DH. If the circles be equal, the point D will evidently be equidistant from A and B. Suppose the radius of the circle about A to be less than the radius of the circle about B, and D must then lie nearer the centre B than A. Wherefore $DG^2 = DH^2$, or (II. 10. El.) $AI^2 - AD^2 = BK^2 - BD^2$, and hence $BK^2 - AI^2 = BD^2 - AD^2$; bisect AB in O, and $BK^2 - AI^2 = AB.2DO$; but the radii AI and BK being both given, the rectangle AB, DO is given, and therefore DO and the point D. Again, because $CE^2 = CF^2$, it follows that $AL^2 - AC^2 = BM^2 - BC^2$ or $BC^2 - AC^2 = BM^2 - AL^2$ or $BD^2 - AD^2$; and consequently CD is at right angles to AB, and given in position.



COMPOSITION.

Bisect AB in O, and make $2AB.OD=BK^2-AI^2$, and erect the perpendicular DC, which is the straight line required.

For (II. 21. El.) $BC^2-AC^2=BD^2-AD^2=2AB.OD$ $=BK^2-AI^2$, and consequently $AI^2-AC^2=BK^2-BC^2$, that is, (II. 10. El.) $CE^2=CF^2$, and $CE=CF$.

Cor. If the circles meet, the straight line CD will either touch at the point of mutual compact, or pass through the points of intersection.

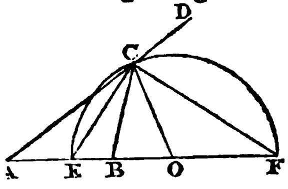
PROP. XXVI. PORISM.

A circle and a point being given, another point may be found, such that straight lines drawn from them to any point in the circumference shall have a ratio which will be given.

The point B may be found, so that AC and BC, inflected to the given circumference ECF, shall have a ratio which may be likewise assigned.

ANALYSIS.

Draw AB, cutting the circle in E and F; join CE, CF; and produce AC. Because E, F are points in the circumference, AC : BC : : AE : EB, and AC : BC : : AF : FB; whence (VI. 10. cor. El.) CE bisects the vertical angle ACB, and CF the adjacent angle BCD, consequently the angle ECF, being the half of both of these, is a right angle, and (III. 19. El.) ECF, a semicircle. Wherefore AF, thus passing through the centre O, is given in position. Now, since AF : FB : : AE : EB, alternately AF : AE : : FB : EB; hence EF, being cut ex-



ternally and internally in the same ratio, EO is (VI. 7. El.) a mean proportional between AO and BO, or $EO^2 = AO.BO$. But AO and EO are given, and therefore BO and the point B are given. Again, because AO : EO : : EO : BO, by division and alternation, AE : EB : : EO : BO; that is, the inflected lines have the given ratio of EO to BO.

COMPOSITION.

Draw AF through the centre of the given circle, and make AO : EO : : EO : BO; B is the point required. For join CO. Because EO is equal to CO, therefore AO : CO : : CO : BO; consequently the triangles ACO and CBO, having besides the common angle at O, are similar, and AC : AO : : BC : CO, or alternately AC : BC : : AO : CO, that is, in a given ratio.

The porism now demonstrated is evidently derived from the local theorem, which forms the 12th Proposition of this Book.

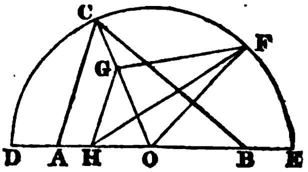
PROP. XXVII. PORISM.

If, from two points in the diameter of a circle equally distant from its centre, straight lines be inflected to the extremity of any arc, another point may be found in the diameter from which a straight line drawn to the termination of the double arc shall contain with the radius a rectangle equivalent to that under the inflected lines.

Let DE be the diameter of a circle having two opposite points, A and B, equally remote from the centre O, and AC and BC be drawn to any point C in the circumference; having made the arc CF equal to DC, a point H may be found in the diameter, such that HF being joined, the rectangle under HF and OD shall be equivalent to the rectangle under AC and BC.

ANALYSIS.

It is obvious, that, in the extreme case, when the arcs DC and DF both vanish, the points C and F must coincide with the extremity of the diameter, and consequently the inflected lines will pass into the segments AD and BD. Wherefore HD.OD = AD.BD; add OA^2 to both, and $HD.OD+OA^2=AD.BD+OA^2=$(II. 17.cor. 2. El.)OD^2 $=HD.OD+OH.OD$: consequently $OA^2 = OH.OD$, and (V. 6. El.) OD : OA : : OA : OH, whence OH and the point H are given. Join OC and OF, draw HG parallel to AC and join GF: Since, by construction, OD : OA : : OA : OH, and (VI. 1. El.) OA : OH : : OC : OG, therefore, by identity of ratios, OD : OA : : OC : OG; but OC being equal to OD, the fourth term OG must (V. 4. El.) be equal to OA. Hence the triangles ACO and GFO, having the sides OC and OA equal to OF and OG, and the contained angle AOC, by hypothesis, equal to GOF, are (I. 3. El.) equivalent; and consequently the base AC is equal to GF. Again, the triangles OCB and GFH are similar (VI. 13. El.); for OC : OA or OB : : OA : OH : : AC or GF : GH, and the contained angle COB is (I. 30. El.) equal to the interior angles OAC and OCA of the triangle ACO, themselves equal to OGF and OGH, which compose the other contained angle FGH. Hence OC : OC



: : FG or AC: HF, and, therefore, (V. 6. El.) HF.BC= AC.BC.

COMPOSITION.

Make OD : OA : : OA : OH, and H is the point required. For double the arc DC to F; join HF and OC, OF; draw HG parallel to AC, and join GF, the triangles ACO and GFO are proved to be equal, and the triangles OCB and GFH similar. The demonstration is indeed exactly the same with that part of the analysis which succeeds the determination of the point H.

Scholium. The equally eccentric points A and B might likewise be taken in the extension of the diameter without the circle, and then the corresponding point H will lie beyond them. When the arc DC is a quadrant, the point F will coincide with E, and the inflected lines become equal. In this case, $AC^2=OD.HE$, which is evident, since (II. 10. El.) $AC^2=OD^2+OA^2=OD^2+OD.OH$. The rectangle under the inflected lines attains a *maximum* likewise in the same situation; for HE being the greatest line that can be drawn from H to the circumference, the rectangle HE, OD, and consequently AC, BC is greatest when the point C terminates a quadrant. There is another *maximum* which belongs to the point directly opposite to C below the diameter. This rectangle AC, BC has also two *minima*, when HF passes into HD in the positions of D and E, the inflected lines being AD, BD, and AE, BE.

If the points A and B fall on the extremities of the diameter, the point H will evidently coincide with D; and consequently the rectangle under the chord of an arc, and the chord of its supplement, is equivalent to the rectangle under the radius and the chord of the double arc, a property demonstrated in the Elements of Trigonometry.

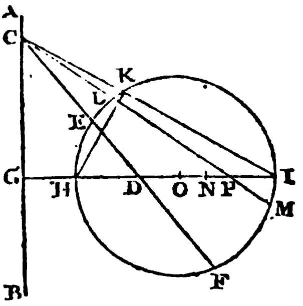
PROP. XXVIII. PORISM.

A circle and a straight line being given in position, a point may be found, such that any straight line drawn from it to the given line, shall be a mean proportional between the segments intercepted by the given circumference.

Let the straight line AB, and the circle HKF be given in position; it is possible to assign a point D, through which a straight line FDC being drawn, CD shall be a mean proportional between the segments CE and CF.

ANALYSIS.

From D let fall upon AB the perpendicular IDG, and join CI and HK. Because CE : CD : : CD : CF, $CD^2 = CE.CF$ = (III. 26. El.) CK.CI; and, since GI passes through the point D, GH : GD : : GD : GI, and $GD^2 = GH.GI$. But (II. 10. El.) $CD^2 = CG^2 + GD^2$, and consequently $CK.CI = CG^2 + GI.GH$; take these away from $CI^2 = CG^2 + GI^2$, and there remains CI.KI = GI.HI. Whence CI : GI : : HI : KI, and consequently the triangles CIG and HIK, having a common vertical angle, are similar. Wherefore the angle HKI, being thus equal to CGI, stands in a semicircle, of which HI is the diameter; con-



sequently GI is given in position, and the points G, H, and I being thence given, the rectangle under GH and GI, or the square of GD, is given, and therefore the point D.

COMPOSITION.

Through the centre O draw the perpendicular GOI; and find (VI. 16. El.) GD a mean proportional to GH and GI; D is the point required. For (III. 26. and II. 17. El.) $CE.CF=CO^2-HO^2=CG^2+GO^2-HO^2=CG^2+GH.GI$; but (V. 6. El.) $GD^2=GH.GI$, and consequently $CE.CF=CG^2+GD^2=CD^2$.

This porism may be supposed to derive its origin from the problem:—" Through a given point P, in the diameter of a circle, to draw a straight line CLPM to the perpendicular AB, so that the rectangle under the segments CL and CM shall be equal to the square of GN." Since (III. 26. El.) $CL.CM=CK.CI=CI^2-CI.KI$; but (II. 10. El.) $CI^2=CG^2+GI^2$, and $CI.KI=GI.HI$; whence $CL.CM=CG^2+GI.GH$, or making $GD^2=GI.GH$, $CL.CM=CG^2+GD^2$ or CD^2, and consequently $CD^2=GN^2$, or $CD=GN$. Wherefore the point D being given, the point C is also given, and thence the straight line CLPM. The problem then is solved by finding GD a mean proportional to GH and GI, and describing, from D with the radius GN, a circle to intersect the perpendicular in C. It is hence evident, that C is independent of the point P. Let CLM, therefore, coincide with CEF, and $CE.CF=GN^2=CD^2$. But this property must evidently obtain, whatever be the position of the point C.

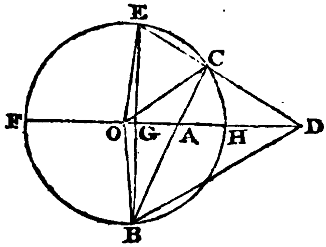
PROP. XXIX. PORISM.

A point being given in the diameter of a given circle, another point in the same extension may be found, such that the angle contained by two straight lines drawn from it to the extremities of a chord passing through the given point, shall be bisected by the diameter.

In the diameter FH of a given circle, let A be a given point through which any chord BAC is drawn; a point D may be found in the extension of the diameter, so that DC and DB being joined, the angle ADC shall be equal to ADB.

ANALYSIS.

Join EB, and draw EO and BO to the centre O. The triangles EOD and BOD, having the side EO equal to BO, OD common, and the angle ODE equal to ODB, and being likewise of the same affection, since the angles DEO and DBO are evidently both acute—are (I. 21. El.) equal, and consesequently the angle EOG is equal to BOG. Whence the triangles OEG and OBG are (I. 3. El.) also equal, and therefore EB is perpendicular to the diameter FH. Wherefore (VI. 9. El.) FA : AH : : FD : DH; but the ratio of FA to AH being given, and consequently that of FD to DH, the point D (VI. 6. El.) is given.



COMPOSITION.

Make (VI. 3. El.) OA : OH : : OH : OD, and then D is the point required. For join OC and OB. Because OH=OC, OA : OC : : OC : OD; wherefore the triangles AOC and COD, having thus the sides about their common angle DOC proportional, are similar; and hence the angle OCA is equal to ODC. In the same manner, it is proved that the angle OBA is equal to ODB. But BOC being an isosceles triangle, the angle OCA is equal to OBA; whence the angle ODC is equal to ODB.

This porism is likewise derived from the local theorem given in Prop. 12. For AC, DC, and AB, DB being inflected in the same ratio, AC : AB : : DC : DB; and consequently (VI. 10. cor. El.) the angle BDC is bisected by DA.

PROP. XXX. PORISM.

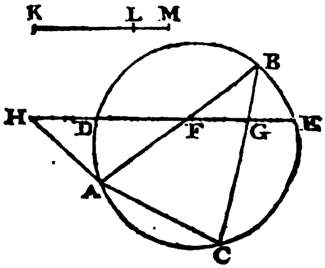
A point being given in the circumference of a circle, another point may be found, so that two straight lines inflected from them to the opposite circumference, shall cut off, on a given chord, extreme segments, whose alternate rectangles shall have a given ratio.

Let the circle ADBE, the point A, and the chord DE, be given in position,—another point C may be found, such that the straight lines AB and CB inflected to the opposite circumference, shall form segments containing rectangles DG, FE, and DF, GE, in the ratio of KM to LM.

ANALYSIS.

Join CA, and produce it to meet the extension of the chord ED in H.

Because KM : LM : : DG.FE : DF.GE, by division, KL : LM : : DG.FE—DF.GE : DF.GE; but DG.FE—DF.GE = (DF + FG) (GE + FG)—DF.GE = FG.DE, and consequently KL : LM : : FG.DE : DF.GE. Make KL : LM : : DE : DH, then KL : LM : : FG.DE : FG.DH; whence FG.DH = DF.GE, and adding DF.FG to both, FH.FG = DF.FE = (III. 26. El.) AF.FB. Wherefore FH : FB : : AF : FG, and (VI. 13. El.) the triangles AFH and GFB are similar, and consequently the angle AHF is equal to FBG; but the angle AHF is given, since the points A, H, and D are given, and, therefore, the chord AC, cutting off from the given circumference a segment that contains a given angle ABC or FBG is given, and thence the point C.



COMPOSITION.

Produce the chord ED to H in the ratio of KM to LM, join HA, and, at any point B in the circumference, make the angle ABC equal to AHF; C is the point required.

For, the triangles AFH and GFB being evidently similar, FH : FB : : AF : FG, and FH.FG = FB.AF = DF.FE; whence FH FG—DF.FG = DF.FE—DF.FG, or FG.DH = DF.GE. But KL : LM : : DE : DH : : FG.DE : FG.DH, and therefore KL : LM : : FG.DE : DF.GE; consequently (V. 9. El.) KM : LM : : FG.DE + DF.GE, or DG.FE : DF.GE.

The porism now investigated arises naturally out of this problem:—" From two given points A and C, one of which lies in a given circumference, to inflect straight lines AB and CB, so as to intercept on the chord DE segments that contain rectangles DG, FE and DF, GE, which are in a given ratio." For, the point H being assumed as before, the analysis requires that the angle ABC should be made equal to AHF. Whence, if on AC, a segment of a circle were described containing that angle, its contact or intersection with the given circumference would determine the point of inflection. Supposing, therefore, the two circles entirely to coincide, the problem will in that case become indeterminate, or admit of innumerable answers.

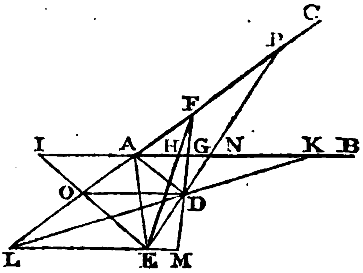
PROP. XXXI. PORISM.

Two points and two diverging lines being given in position, straight lines, inflected from those points to one of the diverging lines, intercept segments, on the other, from points that may be found, and containing a rectangle which will be likewise assignable.

Let DF and EF be inflected, from the points D and E, to the diverging line AC; they will cut off segments, on AB, from points I and K, which may be found, so that the rectangle IH, GK shall be given.

ANALYSIS.

Join EI and EA, DA and DK, and produce ED to meet AC in P. Since A, F, and P are so many points of inflection, it is evident, from the hypothesis, that IA.AK = IH.GK = IN.NK; whence IH : IA :: AK : GK, and, by division, AH : IA :: AG : GK, and alternately AH : AG :: IA : GK. Through E, draw LEM parallel to AB and meeting AC and FD produced; then (VI. 2. El.) LE : LM :: AH : AG :: IA : GK. Again, because IA.AK = IN.NK, IN : IA :: AK : NK, by division AN : IA :: AN : NK, and consequently IA = NK. Wherefore, by substitution, LE : LM :: NK : GK, and LE : EM :: NK : GN, or alternately LE : NK :: EM : GN, that is, (VI. 2. El.) ED : DN; hence (VI. 14. El.) the triangles LDE and KDN are similar, and LDK forms one single straight line. Join DO. Since IA = NK, LE : IA :: LE : NK, that is, (VI. 2. El.) EO : OI :: ED : DN, and therefore (VI. 1. cor. 1. El.) DO is parallel to AB. But the parallels OD and LM being given in position, the points O and L, and thence I and K, are given, and consequently the rectangle IA, AK is given.



COMPOSITION.

Draw DO. EL parallel to AB and meeting the extension of AC; join EO, LD, and produce them to meet AB in I and K; these are the points required. For DF and

EF being inflected, LE : IA : : OE : OI : : ED : DN : : DM : DG : : LM : GK, and alternately LE : LM : : IA : GK; but LE : LM : : AH : AG, and therefore IA : GK : : AH : AG; consequently (V. 8. and 11. El.) IA : IH : : GK : AK, and IA.AK=IH.GK.

The porism thus investigated follows from this problem: "Two straight lines AB and AC being given in position, with the points I and K, E and D, to find a point F, such that the inflected lines EF and DF shall intercept segments IH and GK, containing a given space." For, when the points I and K have the position before assigned, the construction becomes indeterminate.

PROP. XXXII. PORISM.

Three diverging lines being given in position, a fourth may be found, such that straight lines can be drawn intersecting all these and divided by them into proportional segments.

Let AB, CD, and AE be given diverging lines, and HIKL any transverse line cut by them in given ratios; a fourth diverging line FG may be found limiting the segment KL.

ANALYSIS.

Produce EA and GF to meet in M; through K and P draw NO and PO parallel to AB and FG, and meeting in O; join CO; let H′I′K′L′ be another transverse line divided into proportional segments; draw P′I′O′ parallel to PIO and meeting CO in O′, and join O′K′ and produce it to N′.

Because KO is parallel to PH, HI : IK : : PI : IO; and, Now, if the ratio of CA to AF should be the same as that of Y to Z, the point T will coincide with C, and the straight line TO with CO. The problem, therefore, becomes, in this case, porismatic, or every point whatever in CO has the property which belonged before to the single point O.

DEFINITION.

Isoperimetrical figures are such as have equal perimeters, or the same extent of linear boundary.

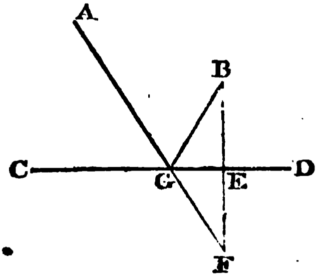
PROP. XXXIII. PROB.

In a straight line given in position, to find a point, whose distances from two given points on the same side shall together be the least possible.

Let it be required, from the points A and B to some point in CD, to draw AG and BG, forming jointly a *minimum.*

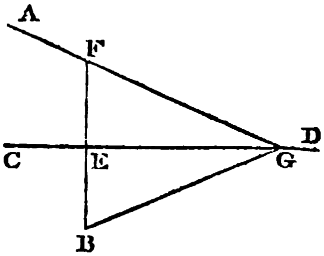
ANALYSIS.

From B, either of the given points, let fall BE a perpendicular upon CD, and, having produced it equally on the opposite side, join GF. It is obvious that the triangles BEG, FEG are equal, and consequently that BG=GF; whence AG + GF is a *minimum.* But the points A and F are evidently both given, and since (I. 15. El.) the shortest communication between them is a straight line, its intersection G with CD is given, and therefore the inflected lines AG and BG are given in position.



Cor It hence appears that, when the combined distance of the points A and B from the straight line CD is the least possible, the incident angles AGC and BGD are equal.

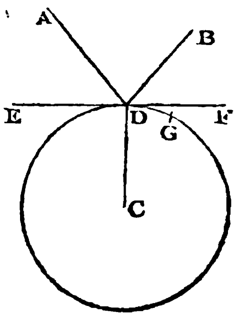
Scholium. Hence also the solution of a similar problem:—"In a straight line given in position, to find a point the difference of whose distances from two given points shall be the greatest possible." If these points lie on the same side of the straight line CD, it is evident that the difference between AG and BG being (I. 15. El.) less than the base AB, this must be the extreme limit, or the difference must reach its *maximum* when AG and BG coincide with AB, and consequently the point G occurs where the production of AB meets CD.—But if A and B lie on opposite sides of CD, let fall the perpendicular BE, which produce till EF be equal to it, and join AF and GF. The triangles BEG, FEG are evidently equal, and therefore BG=GF; but, in the triangle AFG, the difference of AG and GF, being less than AF, must attain its greatest extent, when that triangle is supposed to flatten into a straight line; in which case the angle AGE is equal to BGC.



PROP. XXXIV. THEOR.

Straight lines drawn from two given points to the circumference of a given circle are the least possible, when they make equal angles with a tangent applied at the point of inflection.

Of all the straight lines inflected from the points A and B to the circumference of the circle GDH, AD and BD, which meet the tangent EF at equal angles, form together a *minimum.*



For, by the last proposition, AD and BD, falling at an equal incidence, are jointly shorter than any other lines inflected from the points A and B to the straight line EF; but (I. 16. El.) such lines drawn to that tangent are less than the exterior lines which terminate in the circumference; whence, for both these reasons combined, AD and BD must form the *minimum* of all the straight lines inflected to the circumference GDH.

Cor. It may be shown nearly in the same way, that the lines from A and B obtain their *maximum*, when they are inflected to a point where DC meets the opposite concave circumference.

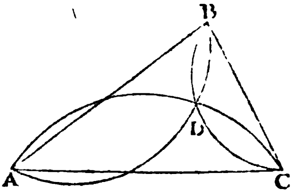
PROP. XXXV. PROB.

To find a point, whose distances from three given points are the least possible.

Let it be required, from the points A, B, and C, to draw AD, BD, and CD, such that their sum shall be a *minimum.*

ANALYSIS.

If the distance BD were supposed to remain constant, the position of D, in the circumference of a circle described from B with the radius BD, must, by the last proposition, be such, when AD and CD together compose a *minimum*, that the angle ADB shall be equal to CDB. For the same reason, if AD continued invariable, BD and CD, completing the *minimum*, must form with it equal angles ADB and ADC. Whence, uniting these conditions, the straight lines AD, BD, and CD all make equal angles about their point of concourse.



COMPOSITION.

Connect the triangle ABC, and upon each of the sides AC and BC describe equilateral triangles, and again circumscribe these by circles, which will intersect in the required point D. For, the angles ADC and CDB, being the supplements of angles of equilateral triangles, are each equal to two-third parts of two right angles, or to one-third of four right angles; consequently three such angles will stand about the point D. Again, while one of the lines AD, BD and DC remains constant, the other two making equal angles with it are the least possible, and consequently the aggregate of all the three is a *minimum*.

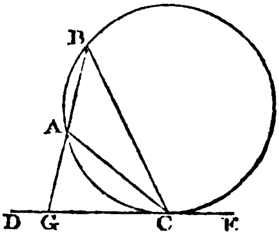
PROP. XXXVI. PROB.

In a straight line given in position, to find a point, at which the straight lines, drawn to two given points on the same side, shall contain the greatest angle.

Let it be required to draw AC and BC, so that the angle ACB shall be a *maximum*.

ANALYSIS.

Describe a circle about the points C, A, and B. Because the angle ACB is greater than any other which has its vertex in DE, the circumference must lie within that straight line, and therefore DE touches the circle.



It is hence evident, (III. 26. cor. El.) that $GA.GB = GC^2$; and, therefore, the point C is assigned, by finding GC a mean proportional to GA and GB.

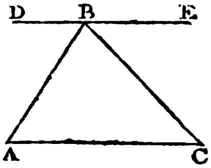
PROP. XXXVII. PROB.

To find a triangle with a given perimeter, and standing on a given base, which shall contain the greatest area.

Let it be required to find a triangle ABC, constituted on the base AC, and containing, within a given perimeter, the greatest possible surface.

ANALYSIS.

Since the base of the triangle ABC is constant while its area forms a *maximum*, the corresponding altitude must evidently be the greatest possible, and consequently the vertex B must lie in a parallel the remotest from AC. Supposing, therefore, the parallel DE to retain its place, the sum of the sides AB and CB, and consequently the whole perimeter of the triangle, will, by the thirty-third Proposition of this Book, be the least possible, when the angle ABD is equal to CBE. Whence, preserving the same perimeter, the parallel will be enabled to recede to the greatest distance from AC, if these incident angles still maintain their equality; but DE being parallel to AC, the alternate angles BAC and BCA (I. 22. El.) are likewise equal, and consequently their opposite sides CB and AB. The triangle ABC is thus isosceles; and it is also given, for its sides are all given.



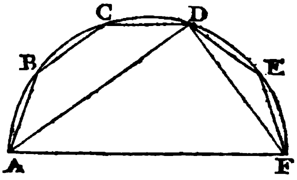
Cor. Hence an equilateral polygon is that which, under a given number of sides, contains, within the same perimeter, the greatest possible surface: For, the rest of the figure remaining constant, suppose any two adjacent sides to vary, and the accrescent triangle so formed will, by this proposition, be a *maximum*, when those sides are equal. The polygon, deriving its expansion from the aggregate of the exterior triangles, must therefore be the greatest possible, when such triangles are in every combination isosceles, and consequently all the sides of the figure equal.

PROP. XXXVIII. THEOR.

If a polygon have all its sides given, except one, it will contain the greatest area, when it can be inscribed in a semicircle, of which that indeterminate side is the diameter.

Let the polygon ABCDEF, having given sides AB, BC, CD, DE and EF, stand upon a base AF, which is variable; the area will attain its *maximum*, when AF becomes the diameter of a circumscribing semicircle.

For, AD and FD being inflected to any point D, the spaces ABCD and DEF will evidently remain the same, while the angle ADF is enlarged, or the points A and F are distended. Whence the polygon must contain the greatest area, when the included triangle ADF contained by given sides AD and DF, is a *maximum*. Now, this will take place when the altitude of the triangle, or the perpendicular let fall from the vertex F upon AD, is the greatest possible. Wherefore (I. 17. El.) ADF is a right angle, and consequently (III. 19. El.) the point D lies in a semicircumference. But the same reason applies to every other intermediate point B, C, or E, of the polygon, which consequently, in its state of *maximum*, is disposed within a semicircle described on the variable side AF.



Cor. 1. Hence a polygon, whose sides are all given, contains the greatest area, when it can be inscribed in a circle. For let ABCD be a polygon, which has each of its sides

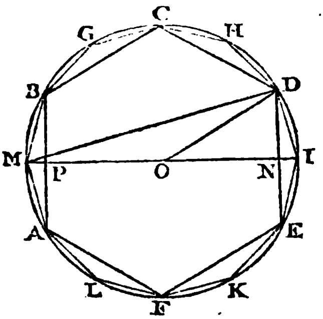
AB, BC, CD, and AD given. Draw the diameter AF, and join DF. The polygon ABCDF is thus a *maximum*; but the triangle ADF being evidently determinate, the remaining polygon ABCD is likewise a *maximum*.

Cor. 2. Hence a regular polygon is that which, with a given perimeter, formed by a given number of sides, contains the greatest area. For, by the corollary to the last Proposition, the sides are all equal; but its angles are (III. 12. cor. 2. and 16. El.) also equal, since it occupies the circumference of a circle.

PROP. XXXIX. THEOR.

A circle contains, within a given perimeter, the greatest possible area.

From the preceding investigations, it appears, that the perimeter and number of sides being given, the figure of greatest capacity is a regular polygon. Let ABCDEF be such a polygon, bounded by the given perimeter: Bisect the corresponding arcs of the circumscribing circle, and another regular polygon MBGCHDIEKFLA will arise, having twice the number of sides. Draw the diameter MI, and join MD and OD. Both polygons are alike composed of triangles equal to ODN and ODI, and consequently the area of the polygon ABCDEF is to that of MBGCHDIEKFLA as ON



to OI, or as 2ON or PN to 2OI or MI. But if this exterior polygon MBGCHDIEKFLA were contracted to the same perimeter with ABCDEF, its area would (VI. 24. El.) be diminished in the ratio of DI^2 to DN^2, that is, (III. 19. and VI. 15. cor. 1. El.) in the ratio of the rectangle MI, NI to MN, NI, or that of MI to MN. Whence (V. 16. El.) the original polygon is to another of equal perimeter and with double the number of sides, as PN to MN. An isoperimetrical figure thus has its area always increased, by doubling the number of its sides. Continuing this duplication, therefore, the regular polygons which arise in succession will have their capacity perpetually enlarged. Whence the circle, as it forms the limit or extreme boundary of all those polygons, must, with a given circumference, contain the greatest possible space.

GEOMETRY

OF

CURVE LINES.

GEOMETRY

OF

CURVES.

ANY straight line produced will either cross another straight line in a single point, or will wholly coincide with it; but the same straight line, in its extension, can meet the bendings of a curve more than once. The number of such possible intersections will, therefore, in every case, serve to discriminate the several orders of lines. A *straight line* itself, since it only admits of being cut once, belongs strictly to the *FIRST ORDER*. A line which is capable of a *double intersection by a straight line* must evidently be incurved, and will range in the *SECOND ORDER*. These lines form, next to the circle, the simplest kind of curves.

BOOK I.

LINES OF THE SECOND ORDER.

GENESIS OF THE CURVE.

The LOCUS *of a point, whose distances from a given point and from a straight line given in position have a given ratio, is a* LINE OF THE SECOND ORDER.

LET F (fig. 1. 2. and 3.) be a given point, and CE a straight line given in position; if another point D so move on the same plane, that its distance DF from F shall have always to the perpendicular DE, or its distance from the line CE, the constant ratio of the given lines X and Y,—it will describe a *Line of the Second Order.*

It is evident, that the describing point will always occupy two corresponding places D and D′, (fig. 9. 10. and 11.) equally distant on either side of the dividing perpendicular CF; for the distances of both these from the given straight line being the same as CH, their distances FD and FD′ from the given point F must be equal, and consequently the right angled triangle HFD is equal to HFD′, and its side DH equal to D′H. Hence the curve consists of two equal similar bends, parted by the extension of the perpendicular CF, which thus forms their axis.

It likewise appears, that the curve must, in every case, cross the perpendicular CF, between the points C and F;

for CF may (VI. 4. El.) be divided at A in any given ratio. But when X is equal to Y, (as in fig. 1.) the curve will not meet that perpendicular again; since evidently no point in the continuation of CF, on either side, can be equally distant from C and F. If X be less than Y, (as in fig. 2.) the curve will again cross the perpendicular CF, on the right hand, at *a*; for (VI. 6. cor. El.) a point may be always found beyond F, such that its distances *a*F and *a*C shall have the given ratio. But if X be greater than Y, (as in fig. 3.) the curve will re-appear, on the left hand of CF, at *a*, where the distance *a*F comes to have to *a*C the given ratio of X to Y.

The generic curve is thus distinguished into three subordinate species: 1. When it stretches out into a single unlimited branch, (fig. 1.); 2. When it returns into itself, and consists of two united and continuous branches, (fig. 2.); and, 3. When it spreads both ways, and forms two opposite diverging branches, extended indefinitely, (fig. 3.)

If the given point F should coincide with C, the *locus* will evidently merge in the perpendicular CF, or (VI. 15. El.) pass into two straight lines, making given angles with CE, and lying obliquely above and below that perpendicular. But if the given straight line CE be supposed to have a situation immeasurably remote from the given point F, the perpendicular distances DE of the describing point D, now comparatively unchanged by its varying position, may be considered as bearing the same ratios; and consequently the distance DF of that point from F, after a very reduced proportion, to be viewed as continuing still unaltered, or the curve becomes transformed into a circle. The double straight line and the circle may thus be regarded as forming the opposite and extreme limits of the lines of the second order. The relations of the circle will hence be found included in the general properties of these curves.

DEFINITIONS.

1. The given or fixed point is termed the *Focus.*
2. The straight line given in position is termed the *Directrix.*
3. The given ratio of the distances of the describing point from the focus and from the directrix is called the *Determining Ratio.*
4. A perpendicular from the directrix, passing through the focus, and extended within the curve, is called the *Axis.*
5. The point where the curve crosses the axis is called the *Vertex.*
6. The middle point of the portion of the axis, limited or intercepted by the vertices of the curve, is called the *Centre.*
7. The distance of the focus from the centre is termed the *Eccentricity.*
8. When the determining ratio is a ratio of *equality,* the curve, consisting of a single extended branch, is named a PARABOLA.
9. When the determining ratio is a ratio of *minority,* the curve, relapsing in a continuous line, is named an ELLIPSE.
10. When the determining ratio is a ratio of *majority,* the curve, spreading on both sides into diverging branches, is named an HYBERBOLA.

Thus, in fig. 1. 2. and 3. F represents the *focus,* CE the *directrix,* and the ratio of X to Y the *determining* ratio; the extended straight line CF is the *axis* of the curve, A or *a* its *vertex,* O (fig. 2. and 3.) the *centre,* and OF, (fig. 2. and 3.) the *eccentricity.* Fig. 1. exhibits the PARABOLA, which has only one vertex A. Fig. 2. exhibits the ELLIPSE, of which the vertices A and *a* lie both on the same side of the directrix: And fig. 3. represents the HYPERBOLA, which has its vertices A and *a* on opposite sides of the directrix.

PROPOSITION I. PROBLEM.

Given the focus, the directrix, and the determining ratio of the curve, to find the termination of its axis.

Let (fig. 1. 2. and 3.) the focus F, the directrix CE and the determining ratio X : Y be given; to find the point A or *a*, where the curve intersects the perpendicular CF, to the directrix, extended through the focus.

ANALYSIS.

The vertex A being a point in the curve, the ratio of AF to AC, its distance from the focus and from the directrix, is the same with the determining ratio, or that of X to Y; but the interval CF is given, and consequently this point A of internal section is given (VI. 6. cor. El.).

If the determining ratio be that of equality, it is obvious that no corresponding external section can exist. But if the determining ratio be not that of equality, the point *a* of external section is likewise given: For (fig. 2. and 3.) *a*F being to *a*C as X to Y, it follows (V. 11. El.) that *a*F is to CF, or the difference between *a*F and *a*C, as X is to its excess or defect from Y; and consequently (VI. 4. El.) the point *a* is given, lying, in the former case, (fig. 2.) on the same side of C with F, and, in the latter, (fig. 3.) on the opposite side.

COMPOSITION.

From the focus draw FG parallel to the directrix, and equal to a fourth proportional to Y, X and CF, the point G will evidently, from the genesis, belong to the curve. On the directrix make C*h* and CH, below and above C, each equal to CF; join G*h* and GH, which being extended will cut the axis in the points A and *a*, the vertices of the curve. For, from the property of diverging lines, FG is to C*h* or CF, that is X is to Y, as AF is to AC, and consequently A is a point in the curve. Again (VI. 2. El.) FG is to CH or CF, that is X is to Y, as *a*F is to *a*C, and therefore *a* is likewise a point in the curve.

In the case of the parabola, (fig. 1.) X being equal to Y, FG is equal to CF or CH, and HG is therefore parallel to CF; whence the second vertex of the curve disappears. In the case of the ellipse, (fig. 2.) X being less than Y, FG is also less than CF or CH, and HG necessarily converges towards the same side of C with the focus. But in the case of the hyperbola, (fig. 3.) X being now greater than Y, FG is likewise greater than CF or CH, and GH meets the axis on the opposite side of the focus.

Cor. 1. Hence in the ellipse and hyperbola, the distance of the focus from the directrix is divided internally and externally in the same determining ratio.

Cor. 2. Hence in the ellipse and hyperbola, the eccentricity is to the semiaxis, and the semiaxis to the distance of the centre from the directrix, in the determining ratio. For since CF is divided by the curve externally and internally in the given ratio, X is to Y (V. 19. cor. 1. El.) as the difference or sum of *a*F and AF, that is 2OF or 2OA, is to the difference or sum of *a*C and AC, that is, 2OA and 2OC, and consequently (V. 3. El.) X : Y : : OF : OA : : OA : OC.

Cor. 3. Hence also in the ellipse and hyperbola, the semiaxis is a mean proportional between the eccentricity and the distance of the centre from the directrix. This follows immediately from the preceding corollary, since OF : OA : : OA : OC.

PROP. II. PROB.

Given the focus, the directrix, and the determining ratio, to find the intersection of the curve with a straight line which is given in position, and parallel to the axis.

The straight line ED, parallel to the axis, being given in position, (fig. 4. 5. and 6.) it is required to find the point D where it meets the curve.

Case I. *When the determining ratio is that of equality.*

ANALYSIS.

Join FD, and from the focus draw FE to the point where the parallel crosses the directrix. Because (fig. 4.) FD is now, by hypothesis, equal to ED, the triangle EDF is isosceles, and consequently (I. 10. El.) the angle EFD is equal to FED; but the straight lines FE and ED being given in position, the angle FED is given, and therefore the equal angle EFD. Whence the position of FD is also given, and consequently its intersection with the parallel ED. Again, since the angles DFE and DEF are both of them acute, and therefore together less than two right angles, FD must (I. 9. El.) always meet ED; but this it can do only once, and hence D is the single point where ED cuts the curve, or where the distances DF and DE from the focus and from the directrix are equal.

Every parallel to the axis will thus meet the parabola, and the curve must therefore spread indefinitely on both sides.

COMPOSITION.

From the vertex A, (fig. 4.) erect AQ perpendicular to CF, and from Q draw QD at right angles to FE, and produce it to meet ED in D; this point is the intersection of the parallel with the curve.

For AF being by hypothesis equal to AC, the side FE of the triangle CFE is (VI. 1. El.) bisected by AQ; and the triangles QDE and QDF, having the side QE equal to QF, QD common to both, and their contained right angles EQD and FQD equal, the base DE is (I. 3. El.) equal to DF, and consequently D is a point in the curve.

CASE II. *When the determining ratio is a ratio of minority or majority.*

ANALYSIS.

FD and FE being joined; because (fig. 5. and 6.) the points F and E are given, and the inflected lines FD and ED have a given ratio of inequality, the locus of the point D is (III. 13. Geom. Anal.) a given circle. But the parallel ED is likewise given in position, and therefore the point or points D and *d*, where it either touches or cuts the circle, (if these lines meet at all,) are hence given.

COMPOSITION.

Join EF, and from the centre O and the vertex A (fig. 5. and 6.) draw OP and AQ perpendicular to the axis, meeting the extension of the oblique line EF in the points P and Q; and from P, as a centre, with the radius PQ, describe a circle touching, if possible, or cutting ED in D and *d*; these are the two points where the parallel ED intersects the curve.

For, join PD and P*d*; and because (Prop. I. cor. 3.)

OF is to OA as OA is to OC, and the diverging lines PE and OC are cut proportionally by the perpendiculars or parallels OP and AQ; therefore PF is to PQ, as the same PQ is to PE; that is, since the radii PQ and PD are equal, as PF to PD, so is PD to PE. Whence the triangles FPD and DPE, having a common angle at P, and their containing sides proportional, are (VI. 13. El.) similar; and, for the same reason, the triangles FP*d* and *d*PE are similar. Consequently FD is to PF, so is DE to PD, and alternately FD is to DE, so is PF to PD or PQ, that is, (VI. 1. El.) as OF to OA. In like manner, F*d* is to PF as *d*E to P*d*, and alternately F*d* is to *d*E as PF to P*d* or PQ, that is, as OF to OA, or in the determining ratio. Wherefore, D and *d* are points in the curve.

In the hyperbola (fig. 6.) the parallel D*d*, lying always between P the centre of the circle and the point Q, must necessarily be cut by the circumference in two points; and hence, as the circle enlarges, the reflected branches of that curve diverge continually.

But in the ellipse (fig. 5.) when PR, the perpendicular to ED, becomes equal to PQ, the circle will touch ED, or this parallel line will meet the curve only in a single point. Beyond this limit, PR will exceed PQ, and consequently the circle will not reach the parallel. The ellipse, therefore, after attaining its extreme width, must again return into itself.

Cor. 1. Hence in the ellipse and hyperbola, a straight line drawn from the centre perpendicular to the axis, bisects that portion of a parallel which is intercepted by the curve. For (fig. 5. and 6.) PR being at right angles to the chord D*d* of the circle, must (III. 4. El.) bisect it.

Cor. 2. When the points D and *d* coalesce at B, the angle PDE, and consequently PFD, becomes each a right angle;

wherefore the adjacent angle BFE is likewise a right angle, and contained in a semicircle; and if a perpendicular were let fall upon its diameter BE, the rectangle under the segments equal and opposite to OF and FC would (III. 26. cor. 1. El.) be equivalent to the square of that perpendicular or OB. But since OF is to OA as OA to OC, the rectangle OF, OC is equivalent to the square of OA, and hence the excess of the square of OA above that of OF, which is the same as the rectangle OF, FC, is equivalent to the square of OB, the lateral limit of the ellipse.

Cor. 3. Hence the angles PFD and PFd are supplemental in the ellipse, and mutually equal in the hyperbola; and the oblique line EF bisects the angle DFd in the hyperbola, and is perpendicular to the bisecting line in the ellipse.

PROP. III. PROB.

The focus, the directrix, and the determining ratio being given, to find the intersection of a line of the second order with a straight line given in position, and perpendicular to the axis.

It is required to find where the perpendicular DHD′ (fig. 9. 10. and 11.) which is given in position, meets the curve.

ANALYSIS.

From the focus, draw FD and FD′ to the points of intersection with the curve. And, because FD is to HC, the distance of the point D or D′ from the directrix, in the determining ratio, and HC itself is given, therefore FD or FD′ is also given, and the locus of D or D′ is a given circle, having F for its centre; but the perpendicular DHD′ being given in position, its intersections with that circle, and consequently with the curve, are likewise given.

COMPOSITION.

Draw the perpendicular FG from the focus, and make it (VI. 3. El.) to FC in the determining ratio; G is evidently a point in the curve. Join CG, and produce it to meet DHD′ in I. From F as a centre, with the radius HI, describe a circle cutting the perpendicular in D and D′; these are the points where that perpendicular intersects the curve.

For (VI. 1. El.) HI or FD is to HC as FG to FC, and consequently FD or FD′ is to HC, or to the distance of the points D and D′ from the directrix, in the given ratio; whence D and D′ are points in the curve.

Cor. 1. It is obvious, from this construction, that when FH becomes equal to IH, the perpendicular DHD′ will meet the curve in a single point only; and that if FH should exceed IH, it will not meet the curve at all. But when FH becomes equal to IH, the angle HFI is equal to HIF, and each of them is consequently half a right angle. To define the limits of intersection, therefore, draw FL and F*l* on either side at half right angles with CO, and from the points L and *l*, where they meet the oblique line CK, let fall perpendiculars LA and *la*, which will limit the transverse axis.

Cor. 2. In the parabola (fig. 9.) FG is always equal to FC, and the obliquity FCG is *equal* to half a right angle; in the ellipse, (fig. 10.) FG is less than FC, and FCG is *less* than half a right angle; but in the hyperbola (fig. 11.) FG being greater than FC, the obliquity FCG is *greater* than half a right angle. Wherefore in the parabola, F*l* is parallel to CK, and never meets it, or the curve stretches out indefinitely. In the ellipse, F*l* converges to CK on the same side with the focus F; but in the hyperbola, it meets that boundary on the opposite side, or the curve reappears in a reflex position.

Cor. 3. The oblique line CK (fig. 9. 10. and 11.) is a tangent to the curve at G, and the intercepted perpendicular OK from the centre is equal to half the axis OA. For HD, being less than the hypotenuse FD, is consequently less than HI, or every point except G lies within CK. Again, OK is to OC as FG to FC, that is, in the determining ratio, or (Prop. I. cor. 2.) as OA to OC; whence OK is equal to OA.

PROP. IV. PROB.

Given the focus, the directrix, and the determining ratio, to find the intersection of the curve with a straight line drawn in a given position through the focus.

Let (fig. 17. 18. 19. and 20.) MF, passing through the focus F, be given in position; it is required to find where that straight line meets the curve.

ANALYSIS.

From the point of intersection D or *d*, let fall the perpendicular DE or *de* upon the directrix: And because DE and DM are both of them given in position, their mutual inclination, or the angle EDM, is given, and consequently the right-angled triangle DEM is given in species. Wherefore the ratio of DE to DM is given; and the ratio of FD to DE being given from the genesis of the curve, the compound ratio of FD to DM, or that of F*d* to *d*M, is likewise given. The distance FM, being thus divided internally and externally in the same given ratio, its points of section D and *d* are (VI. 6. 11. El.) hence given.

COMPOSITION.

Through the focus F draw GFG′ perpendicular to the axis; make FG, or FG′, to FC in the determining ratio; from the directrix cut off on either side MN or MN′, equal to MF; and join NG or N′G, producing them to meet MF, or its extension, in D or *d*; these are the points of intersection required.

For, from the property of parallel and diverging lines (VI. 2. El.), FD is to DM, or F*d* is to *d*M, as FG to MN′ or MN, that is MF; but, for the same reason, DM is to DE, or *d*M is to *de*, as MF to FC, and consequently, by composition of ratios, FD is to DE, or F*d* is to *de*, as FG to FC, that is in the determining ratio. The points D and *d* are therefore in the curve.

Cor. Hence the lines NF and N′F bisect the exterior angles MFG′ and MFG. For MF, being equal to MN or MN′, the triangles MFN and MFN′ are isosceles, and consequently the angle MFN is equal to MNF or the alternate angle NFG, and MFN′ equal to MN′F or N′FG.

Scholium. Since the straight lines MF and N′G cross each other, the nearer intersection D must obtain in every case; but the point *d* of remote intersection is subject to a variety of conditions.

1. In the parabola (fig. 17.) the oblique line MF is (I. 18. El.) greater than the perpendicular FC or FG, and consequently NG and MF must always converge to some point

d beyond the focus. Hence, though the curve continually bends from the axis, its extreme divarication is yet less than any assignable angle.

2. In the ellipse (fig. 18.) MF being greater than FC, is, from the nature of the curve, still greater than FG, and therefore NG and MF must always meet when produced. The ellipse hence entirely encompasses its focus.

3. In the hyperbola (fig. 19. 20. and 21.) since FG is greater than FC, the oblique distance MF, though greater also than FC, may yet be greater, equal, or less than FG; the straight line NG may therefore converge on MF (fig. 19.), run parallel to it (fig. 20.), or diverge from it (fig. 21.), and consequently meet on the opposite side. Hence, the divarication of the hyperbola from its axis exceeds not a certain limited inclination.

DEFINITIONS.

11. The part of the axis intercepted by the curve being named the *transverse axis*, the portion of the perpendicular passing through the centre, and terminated on both sides by the curve, is called the *conjugate axis*.

12. A straight line which bears to the distance of any point within or without the curve from the directrix the determining ratio, is called, in reference to that point, the *Apportionate Distance*.

13. The distance, whether real or apportionate, of the focus from any point has to an oblique line drawn from that point at a given inclination to the directrix, a ratio which is termed the *Deflected Ratio*.

Thus, in the ellipse and hyperbola, A*a* (fig. 13. and 14.) represents the *transverse axis*, connecting the reverted branches of the former curve and the divergent branches of the latter. In the ellipse, B*b* expresses the *conjugate axis* or the extreme width; but in the hyperbola, this conjugate axis can strictly have no existence, though by observing the due change of relations, it is represented by an *imaginary line* commonly named the *secondary axis*.

Again, if the straight line KL (fig. 22. 23. and 24.) be to a perpendicular let fall from K upon the directrix in the ratio of FA to AC, it is called the *apportionate distance* of the point K; and if MD*d* makes a given angle with the directrix, the ratio of KL to KM, or of DF to DM, is named the *deflected ratio* of that oblique line.

PROP. V. PROB.

The focus, the directrix, and the determining ratio being given, to find the intersection of the curve with a straight line drawn from a given point in the directrix through the centre.

Let the straight line MO (fig. 7. and 8.) be extended from the point M in the directrix through the centre O, to discover where it cuts or meets the curve.

ANALYSIS.

Join FM, FD and FD′. Since MO is evidently given in position, the deflected ratio of FD to DM, or of FD′ to D′M is given; but FM is given both in magnitude and position, and consequently the angle FMD or FMD′ is given. Wherefore the triangle FDM or FD′M having given the side FM, the angle FMD, and the ratio of the sides about the other angle at D or D′, are (VI. 14. El.) given, and the points D and D′ are hence given.

COMPOSITION.

Upon the transverse axis A*a*, describe a circle crossing the extension of MF in the points L and L′; join OL and OL′, and parallel to them draw FD and FD′, from the focus, to meet MO in the points D and D′; these are the intersections required.

For, from the property of parallel and diverging lines, as OM to DM, so is OL to FD, or alternately, as FD to DM, so is OL or OA to OM. But the apportionate distance of the centre O is evidently OA, since the deflected ratio corresponding to the position MDOD′, being compounded of the ratio OA to OC, and of that of OC to OM, is the same as the ratio of OA to OM. Wherefore FD is to DM in that deflected ratio, and D is hence a point in the curve. In like manner, as OM to DM, so is OL′ to FD′; and alternately, as FD′ to D′M, so is OL′ or OA to OM. Consequently FD′ is to D′M in the same deflected ratio, and D′ is another point in the curve.

Cor. 1. Hence every straight line drawn through the curve is bisected in the centre. For, since OF : OA : : OA : OC, the straight line LL′ is (III. 2. An.) cut externally in M and internally in F in the same ratio, or ML : ML′ : : FL : FL′; whence alternately ML : FL : : ML′ : FL′; but (VI. 1. El.) ML : FL : : MO : OD, and ML′ : FL′ : : MO : OD′, consequently MO : OD : : MO : OD′, and therefore (V. 4. El.) OD is equal to OD′.

Cor. 2. Hence the transverse axis A*a* is in the ellipse (fig. 7.) equal to the sum, and in the hyperbola (fig. 8.) to the difference, of the lines FD, FD′ drawn from the focus to the extremities of any straight line DOD′, passing through the centre and terminating in the curve. For $OA : OM :: FD' : MD' :: FD : MD$, and consequently (V. 19. El.) $OA : OM :: FD' \pm FD : MD' \pm MD$; but $2OM = MD' + MD$ in the ellipse, and $2OM = MD' - MD$ in the hyperbola. Wherefore (V. 4. El.) $2OA$ or Aa is equal to $FD' + FD$ in the ellipse, and to $FD' - FD$ in the hyperbola.

Cor. 3. Hence in the ellipse and hyperbola, the inflected lines FD and FD′ make equal angles either on the same or on opposite sides with the focal line FM. For LOL′ being evidently isosceles, the angle OLL′ (I. 10 El.) is equal to OL′L; but (I. 22. El.) OLL′ is equal to DFM and OL′L equal to D′FM; whence the angle DFM is equal to D′FL.

Cor. 4. Hence, in the ellipse and hyperbola, the curve bears the same relation to another focus and directrix assumed at corresponding distances on the other side of the centre. For let O*f* and O*c* be made equal to OF and OC, produce MO to meet the perpendicular *cm* in *m*, and join *f*D and *f*D′. The triangles FOD and *f*OD′ are equal, since they have the sides OF and OD equal to O*f* and OD′ and their contained angles equal; whence (I. 3. El.) the base FD is equal to *f*D′. But OM being equal to O*m*, MD is obviously equal to *m*D′, and therefore *f* D′ is to *m*D′ in the deflected ratio; whence *f*D′ is to the distance of the point D′ from the directrix *cm* in the determining ratio.

Scholium. It is evident that the focal line MF must always cut the circle about the ellipse (fig. 7.) in two points, and consequently MO will form two intersections. But this line MF (fig. 8.) may either cut the circle within the hyperbola, or touch it, or lie altogether without it. When MF comes to touch the circle, the points L and L′ coalesce, and the corresponding lines FD′ and FD extend themselves in opposite directions parallel to DOD′, which therefore they cannot meet. In this position, the obliquity of DOD′, in respect to the axis, serves to limit the divergency of each branch of the curve.

If we conceive the centre O to be removed to an indefinite distance, the portion of the circle which cuts the focal line MF will merge in a tangent or perpendicular at A, and consequently the intersection L′ and the lines OL′, FD′, and MD′ will disappear, while the corresponding lines OL and MD all become parallel to the axis. Hence, in the parabola, a line which should tend to the centre is represented by a parallel to the axis.

PROP. VI. THEOR.

If from any point in the curve, two straight lines be drawn to the foci, their sum in the ellipse, and their difference in the hyperbola, is equal to the transverse axis.

Let (fig. 7. and 8.) DF and D*f* be drawn to the two foci, their sum is equal to A*a* in the ellipse (fig. 7.) and their difference is equal to A*a* in the hyperbola (fig. 8.).

For join FD and *f*D′. The triangles FOD′ and *f*OD are equal, since they have the side OF equal to O*f*, the side OD′ (by the first corollary to the last proposition) equal to OD, and a common vertical angle; whence the base FD′ is equal to *f*D′, and the sum or difference of FD and FD′ is equal to the sum or difference of FD and *f*D. But (by the second corollary to the same proposition) the transverse axis A*a* is equal to the former; wherefore in the ellipse, the transverse axis (fig. 7.) is equal to the sum of *f*D and FD; and in the hyperbola (fig. 8.) the transverse axis is equal to the excess of *f*D above FD.

Cor. 1. Hence in the ellipse a straight line drawn from either of the foci to the extremity of the conjugate axis is equal to the semitransverse axis. For the right-angled triangles FOB and *f*OB, having the side OF equal to O*f*, and OB common, are (I. 3. El.) equal; and consequently FB is equal to *f*B, and each of them equal to half their sum, or to half the transverse axis A*a*.

Cor. 2. Hence, in the ellipse and hyperbola, if the transverse axis A*a* be divided into segments AK, *a*K, equal to straight lines FD, *f*D drawn from the curve to the foci, the distances OH, OK of the centre from the perpendicular let fall upon the axis, and from the point of section, are in the determining ratio. For AK being made equal to FD, it is evident that *a*K will be equal to *f*D; consequently (II. 21. El.) $fD^2 - FD^2$, or $aK^2 - AK^2 = 2OF.2OH =$ (II. 17. El.) 2OA.2OK, and OF.OH=OA.OK; wherefore (V. 6. El.) OK : OH :: OF : OA.

Cor. 3. In the ellipse, the square of the semiconjugate axis is equivalent to the excess of the square of the semitransverse above the square of the eccentricity, or to the respective rectangles under the segments into which the transverse axis and the distance of the centre from the directrix are divided by the focus. For, FB being equal to OA, it follows (II. 10. El.) that FB^2 or $OA^2 = OB^2 + OF^2$, and $OB^2 = OA^2 - OF^2$; but $OA^2 - OF^2 =$ (II. 17. El.) $AFa = COF - OF^2 = OFC$.

Scholium. The analogy of the hyperbola to the ellipse may be preserved, by conceiving a *secondary* semiaxis, whose square is still the difference between the squares of the eccentricity and of the semitransverse axis; but, in this case, it is the *defect* of the square of the semitransverse from the square of the eccentricity. Thus, if AB′ (fig. 8.) or A*b*′, equal to the eccentricity OF, be inflected from the vertex A of the curve to the perpendicular axis, they will define the *imaginary* or *secondary* axis. From this construction, it is obvious, that $Ob^{2\prime}$ or $OB^{2\prime} = OF^2 - OA^2$.

But the square of this imaginary semiconjugate axis is likewise equivalent, as in the ellipse, to the rectangles under the segments of the transverse axis and of the distance of the centre from the directrix, as divided externally by the focus. For $OB'^2 = OF^2 - OA^2 =$ (II. 17. El.) AFa, and again OB'^2 or $OF^2 - OA^2 =$ (Prop. I. cor. 2.) $OF^2 - COF = OFC$. Hence the semiconjugate axis of the hyperbola may be determined from the intersection S of the directrix with a semicircle described on the transverse axis Aa, or with a semicircle described on OF the eccentricity; FS being a tangent to the former, and a chord in the latter, and consequently (III. 26. cor. 1. and 2. El.) $FS^2 = AFa = OFC$. The right-angled triangle OSF in fig. 8. is thus to be viewed as representing BOF in fig. 7. It is also demonstrated, as in the last corollary, that $OD^2 = OA^2 - OF^2 + OK^2 = OK^2 - OB'^2$, and consequently that OD is, in the hyperbola, equal to a tangent drawn from K to a circle described from O with the radius OB′.

PROP. VII. THEOR.

If the transverse axis of an ellipse or hyperbola be divided into segments equal to lines drawn from the foci to any point in the curve, the square of its distance from the centre will be equivalent to the sum or difference of the squares of the semiconjugate axis, and the distance of intermediate section from the centre.

Let the transverse axis A*a* (fig. 7. and 8.) be cut in K, so that AK and *a*K shall be equal to the focal lines FD and *f*D; then in the ellipse (fig. 7.) $OD^2 = OK'^2 + OB^2$, but in the hyperbola (fig. 8.) $OD^2 = OK^2 - OB^2$.

For the base of the triangle FD*f* being bisected in O, twice the squares of OD and OF are (II. 25.) equivalent to the squares of FD and *f*D, or the squares of AK and *a*K, which again (II. 18. El.) are equivalent to twice the squares of OA and OK; whence the squares of OD and OF are together equivalent to the squares of OA and OK. But in the ellipse (fig. 7.) the square of OA being equivalent to the squares of OF and OB, the squares of OD and OF are equivalent to the three squares of OF, OB and OK; and, taking the square of OF from both, there remains the square of OD equivalent to the squares of OB and OK. In the hyperbola (fig. 8.) the square of OF being equivalent to the squares of OA and OB′, the three squares of OD, OA and OB′ must be equivalent to the squares of OA and OK, and consequently the squares of OD and OB′ are equivalent to the square of OK, or the square of OD is equivalent to the excess of the square of OK above that of OB′.

Cor. 1. Hence, in the ellipse, (fig. 7.) OD, the distance of the centre from any point in the curve, is equal to BK, the distance of the extremity of the conjugate axis from the point of intermediate section. For, by the proposition, $OD^2 = OB^2 + OK^2 =$ (II. 10. El.) BK^2, and consequently $OD = BK$.

Cor. 2. Hence, in the ellipse, (fig. 7.) $BF^2 - BK^2$ (II. 21. cor. El.) $= OF^2 - OK^2$, or $OA^2 - OD^2 = OF^2 - OK^2$ $=$ (II. 17. El.) FKf, and consequently $OD^2 = OA^2 - FKf$. But, in the hyperbola, (fig. 14.) $OD^2 = OK^2 - OB'^2 = OK^2 - OF^2 + OA^2 = OA^2 + FKf$.

Cor. 3. In both the ellipse and hyperbola (fig. 7. and 8.) $DH^2 = AFa - HKh$. For, in the ellipse, $OD^2 = OB^2 + OK^2$, and $DH^2 = OD^2 - OH^2 = OB^2 - OH^2 + OK^2 = OB^2 - HKh = AFa - HKh$. But, in the hyperbola, $OD^2 = OK^2 - OB'^2$, and $DH^2 = OH^2 - OK^2 - OB^2 = HKh - OB^2 = HKh - AFa$.

Cor. 4. In both the ellipse and hyperbola (fig. 7. and 8.) $DH^2 = AHa \backsim FKf$. For in the ellipse $OD^2 = OA^2 - FKf$, and $OD^2 - OH^2 = OA^2 - OH^2 - FKf$; that is, $DH^2 = AHa - FKf$. But in the hyperbola $OD^2 = OA^2 + FKf$, or $OD^2 - OH^2 = OA^2 - OH^2 + FKf$; that is, $DH^2 = FKf - AHa$.

DEFINITIONS.

14. A focal chord perpendicular to the transverse axis, is called the *Latus Rectum* or *Parameter*.

15. A straight line cutting an axis at right angles and terminating in the curve, is called an *Ordinate*, and either segment of that axis which it intercepts is named an *Absciss*.

Thus, in fig. 12. 13. and 14., the line GFG′, drawn at right angles through the focus, is the *Parameter of the Axis*; and the perpendicular DH is an *Ordinate* to the transverse axis, forming the *Absciss* AH in the parabola, and AH or H*a* in the ellipse and hyperbola; while DS in these two curves is an *Ordinate* to the conjugate or secondary axis, which it divides into the *Abscissæ* DS and S*d*.

In the parabola, FG being equal to CF, the parameter GG′ is consequently equal to twice CF or four times AF.

PROP. VIII. THEOR.

In the parabola, the square of an ordinate applied to the axis is equivalent to the rectangle under the absciss and the parameter; but, in the ellipse and hyperbola, the square of an ordinate is a fourth proportional to the transverse axis, its parameter, and the rectangle under the segments of that axis.

1. *In the Parabola.*

Let DH (fig. 12.) be perpendicular to the axis, from which it cuts off a portion AH; the square of this ordinate DH will be equivalent to the rectangle under the parameter and the absciss AH.

For join FD. In the right angled triangle DHF, the square of the hypotenuse FD is (II. 10. El.) equivalent to the squares of DH and FH; or the square DH is equivalent to the difference of the squares of DF and FH, and consequently (II. 19. El.) to the rectangle under the sum and difference of DF and FH. But, from the nature of the curve, DF is equal to DE or HC, and AF equal to AC; whence the sum of DF and FH is equal to the double of AH, and their difference equal to CF. Wherefore the square of DH is equal to double the rectangle under CF and AH, or the rectangle under AH and the double of CF, which is the parameter of the axis.

2. *In the Ellipse and Hyperbola.*

Let DH (fig. 13. and 14.) be perpendicular to the axis which it cuts internally or externally, in two portions AH and H*a*; as the transverse A*a* is to its parameter, so is the rectangle under the segments AH, H*a*, to the square of DH.

For, by the second corollary to the sixth proposition, $OA : OF :: OH : OK$, and (V. 22. cor. 1. El.) $OA^2 : OF^2 :: OH^2 : OK^2$, and, by conversion, $OA^2 : OA^2 \backsim OF^2 :: OH^2 : OH \backsim OK^2$; but, from the third corollary to the same proposition, the difference between the squares of OA and OF is equivalent to the square of OB or OB′, the conjugate or the secondary axis, and (II. 17. cor. 1. and 2. El.) the difference between the squares of OH and OK is equivalent to the rectangle HK*h*; whence $OA^2 : OB^2$ or $OB'^2 :: OH^2 : HKh$. Again (V. 19. cor. 1. El.) $OA^2 : OB^2$ or $OB'^2 :: OA^2 \backsim OH^2 : OB^2 \backsim HKh$; the difference, however, between the squares of OA and OH is, in both cases, equivalent to the rectangle AH, H*a*; and in the ellipse (fig. 13.) by Prop. VII. cor. 3. $DH^2 = OB^2 \backsim HKh$. Wherefore, by substitution, $OA^2 : OB^2 :: AHa : DH^2$. Hence, when the ordinate passes through the focus, $OA^2 : OB^2 :: AFa$ or $OB^2 : FG^2$; and consequently (V. 24. El.) $OA^2 : OB^2 :: OA : FG$; wherefore $OA : FG$, or $Aa : GG' :: AHa : DH^2$.

In the hyperbola, (fig. 14.) it was also shown in the same corollary, that $DH^2 = OB'^2 \backsim HKh$, and hence $OA^2 : OB'^2 :: AHa : DH^2$; wherefore likewise $Aa : GG' :: AHa : DH^2$.

Cor. 1. Hence conversely, the square of the conjugate or secondary axis is to the square of the transverse, as the rectangle under the segments of the former, or the difference between the square of the semiconjugate and of the distance of the centre from the point of section in the ellipse, or the sum of those squares in the hyperbola, is to the square of an ordinate applied to the conjugate or secondary axis. For (fig. 13. and 14.) $OA^2 : OB^2$ or $OB'^2 ::$

$AHa : DH^2$, and (V. 19. cor. 1. El.), $OA^2 : OB^2 :: OA^2 - AHa : OB^2 - DH^2$ (fig. 13.), or $OA^2 : OB'^2 :: OA^2 + AHa : OB'^2 + DH^2$ (fig. 14.); but $SD^2 = OH^2 =$ (II. 17. cor. 1. El.) $OA^2 - AHa$ (fig. 13.) or $OA^2 + AHa$ (fig. 14.) and $OB^2 - DH^2$ (fig. 13.) $= BSb$, whence (fig. 13.) $Bb^2 : Aa^2 :: BSb : SD^2$, and (fig. 14.) $B'b'^2 : Aa^2 :: OB'^2 + OS^2 : SD^2$.

Cor. 2. Hence ordinates or perpendiculars to both axes equally distant from the centre are equal; for the segments of those axes are then equal.

Cor. 3. Hence if a circle be described on the transverse axis, a perpendicular or a tangent drawn to it from a point of internal or external section will be proportional to the corresponding ordinate in the ellipse or hyperbola. For (fig. 15.) $MH^2 =$ (III. 26. cor. 1. El.) AHa, and consequently $Aa^2 : Bb^2 :: MH^2 : DH^2$, or $Aa : Bb :: MH : DH$, and hence the perpendicular HM is to the ordinate HD in the constant ratio of the two axes. Again, because (III. 26. cor. 2. El.) $MH'^2 = AH'a$, it follows that the tangent MH′ is to the ordinate DH′ of the hyperbola, likewise in the constant ratio of its axes.

Scholium. From the property of the ellipse, that its ordinates are proportional to the perpendiculars of the circumscribing circle, is derived the principle of the *elliptical compasses.* On the transverse axis Aa (fig. 16.) let a circle be described; produce the ordinate DH or D′H, to meet the circumference in M; join OM, and, parallel to it, draw DTV or T′D′V′, cutting the two axes in T and V, or T′ and V′. Because, by the last corollary, $ON : OB :: MH : DH$, and (VI. 2. El.) $MH : DH :: OM : TD$, it follows that ON is to OB, as OM, which is equal to VD or V′D′, to TD or T′D′; and hence VD and TD, or V′D′ and T′D′, are equal to the two semiaxes ON and OB. If therefore DT, or D′T′, be assumed equal to the semiconjugate axis OB, and DV on the same side, or D′V′ on the opposite side, be made equal to the semitransverse OA, and if the inflexible line VTD or V′D′T′ be turned round, carrying the points V, T, or V′, T′, along the axes, the external point D, or the internal point D′, will either of them describe the ellipse.

When the axes of the ellipse become equal, the curve passes into a circle, corresponding to which is the variety of hyperbola, called the *equilateral* or *rectangular*, the square of its ordinate being equivalent to the rectangle under the abscisses. In fig. 58., H′D = H′M, and $hD^2 =$ (II. 17. cor. 2.) $Oh^2 + OB^2 =$ (II. 10. El.) hA^2, or the secondary ordinate hD, is always equal to the hypotenuse hA.

If the parabola be viewed as the limiting curve between the ellipse and the hyperbola, when their transverse axis becomes indefinitely great, its property may be comprehended under the general demonstration. For $Aa : GG' : : AH.Ha : DH^2$, or (V. 3. El.) $Aa.AH : GG'.AH : : AH.Ha : DH^2$, and alternately $Aa.AH : AH.Ha : : GG'.AH : DH^2$. But the ratio of Aa to Ha approaches to equality, and consequently so does the ratio of $Aa.AH$ to $AH.Ha$, or that of $GG'.AH$ to DH^2. Wherefore ultimately $GG'.AH = 4AF.AH = DH^2$.

PROP. IX. PROB.

The focus, the directrix, and the determining ratio being given, to find the intersection of the curve with an oblique line given in position.

Let the straight line MD*d* (fig. 22. 23. and 24.), be drawn from a given point M in the directrix at a given angle; it is required to find where it meets the curve.

ANALYSIS.

Suppose this oblique line to cut the curve in the points D and *d*, and join FD, F*d*, and FM. Because MD is given in position, the deflected ratio corresponding to its inclination is given (VI. 2. El.); but MF being given both in position and magnitude, the triangle MFD has consequently an angle at M, with the adjacent side MF given, and the sides FD and DM containing another angle in a given ratio, and is therefore given (I. 21. El.) when its character as acute, obtuse, or right angled, is known.

If the remaining angle DFM (fig. 22. 23. and 24.) be a right angle, the triangle will have a single aspect: but if that angle be oblique, the triangle will show a double form, being either acute or obtuse, and having the angles MFD and MF*d* mutually supplemental. When the oblique line meets the opposite branches of the hyperbola (fig. 25.), the triangles MFD and MF*d*, instead of having the same vertical angle at M, will have their vertical angles FMD and FM*d* supplemental. In this case, it is obvious, that only one form of triangle can exist on each side of FM.

COMPOSITION.

In MD take any point K, from which as a centre, and with a radius equal to the apportionate distance, describe a circle meeting MF in L and *l*; join KL and K*l*; and parallel to them, draw FD and F*d* from the focus to meet MD in the points D and *d*; these are points in the curve.

For, from the property of parallel and diverging lines, FD is to DM, or F*d* to *d*M, as LK or *l*K to KM, that is, in the deflected ratio; and consequently D and *d* are points in the curve.

Cor. 1. Hence the lines FD and F*d* drawn from the focus to the points of section, are equally inclined on either side of MF, the straight line joining it with the point in the directrix. For (fig. 22. 23. and 24.) the angle DFM = (I. 22. El.) KL*l* = (I. 10. El.) K*l*L = *d*FN; and in the case of opposite section (fig. 25.) DFM = KL*l* = K*l*L = MF*d*.

Cor. 2. Hence, having produced *d*F to meet the directrix in P, the line MF bisects the angle DFP; for *d*FN, which is vertical to MFP, is equal to MFD.

Cor. 3. If the acute angles DFM and *d*FN be supposed to increase (fig. 22. 23. and 24.), and the lines FD and F*d* mutually to approach till they come to coincide, as in fig. 26. 27. and 28.; then, while MF remains constant, the angle FMD will gradually augment; the secant MD*d* now passes into a tangent MT, and a line FT, joining the focus with the single point of contact T, becomes perpendicular to MF. In the opposite branches of the hyperbola, (fig. 28.) when the adjacent angles FMD and FM*d* are right angles, the triangles MFD and MF*d* evidently (I. 21. El.) become equal, and the points of section D and *d* appear at equal distances on opposite sides of M; and if the equal angles DFM and *d*FM become at the same time right angles, no section will take place, but the points D and *d*, instead of approaching, will separate to the utmost, and entirely vanish away.

Cor. 4. Hence the tangents TM, T′M (fig. 26. 27. and 28.) applied at the extremities of a focal chord TT′, make at their concourse M a right angle in the parabola, an acute angle in the ellipse, and an obtuse angle in the hyperbola. For the perpendiculars TN, T′N′ being let fall upon the directrix, the right-angled triangles TFM, TNM, having a common hypotenuse, and the side FT, equal to TN in fig. 26., less than it in fig. 27., and greater than it in fig. 28., the angle FMT is (III. 12. and 16. El.) equal to NMT in fig. 26., less than it in fig. 27., and greater than that angle in fig. 28. For the same reason, the angle FMT′ is, in these successive figures, equal, less, or greater than N′MT′. Consequently the compound angle TMT′, being thus respectively equal, less, or greater than its adjacent angles TMN and T′MN′, is right-angled in the parabola, acute in the ellipse, and obtuse in the hyperbola.

Scholium. The different phases exhibited by the concourse of a straight line with the curve, are all easily derived from the relation of the points L and *l* to each other and to the point M. The points of section D and *d* occur on the same side or on opposite sides of the directrix, according to the situation of the corresponding points L and *l* in respect to M. When L and *l* coincide, KL, and consequently its parallel DF, becomes perpendicular to MF, and the secant merges into a tangent; but, when L and *l* concentrate in M, the straight line FD, being now parallel to DM, will never meet it, and the points D and *d* will vanish in opposite directions. Since the oblique distance MK of the assumed point K from the directrix is greater than its perpendicular distance, it is evident that the apportionate focal distance KL is always less than MK in the parabola, and still more so in the ellipse, but may be less, equal, or greater in the hyperbola. The points L and *l* will therefore lie on the same side of M in the parabola and ellipse, (fig. 22. and 23.), and according to the degree of the inclination of DM will, in the hyperbola (fig. 24. and 25.) lie on the same side of M, coalesce with it, or appear on opposite sides; and hence the sections of the oblique line DM with the parabola and ellipse, if it meets the curve, will take place on the same side of the directrix, but it will always cut the hyperbola either on the same or opposite sides, except in the limit of transition.

DEFINITION.

16. Any straight line drawn through the centre to terminate both ways in the curve, which in the parabola is represented by a parallel to the axis, is called a *Diameter*.

Thus in fig. 34., TH, a parallel to the axis, is a *diameter* of the parabola; and, in fig. 35. and 36., TO*t* drawn through the centre is a *diameter* to the ellipse or hyperbola.

PROP. X. THEOR.

If through any point of the curve, a diameter be produced to meet the directrix, a perpendicular let fall from that point, upon the straight line joining the intersection of the directrix with the focus, will be a tangent.

Let a diameter be extended through the point T, (fig. 34. 35. and 36.) to meet the directrix in M, and this point be joined with the focus, a straight line TL or *tl* drawn at right angles to MF, will touch the curve at T or *t*.

1. *In the Parabola.*

Produce (fig. 34.) TL to meet the directrix in Z, and join FT and FZ. The triangles LTF and LTM, having the side TF equal to TM, the side TL common, and the adjacent angles at L right angles, are equal (I. 21. El.), and therefore the angle FTL or FTZ is equal to MTL or MTZ. Again, the triangles FZT and MZT, having thus the angles at T equal, and their containing sides, are equal (I. 3. El.); wherefore the angle TFZ is equal to TMZ, and hence a right angle. Consequently, by Cor. 3. to the last Proposition, TZ touches the curve at T.

2. *In the Ellipse and Hyperbola.*

Produce (fig. 35. and 36.) TL to the directrix at Z, draw OP parallel to TZ, extend MF to meet it in I, and join FT and FZ. From the property of parallel and diverging lines, TL : TZ : : OI : OP, and consequently (V. 13. El.) TL.TZ : TZ^2 : : OI.OP : OP^2. But the right angled triangles PCF and PIF, standing on the same base PF, are contained in a circle; wherefore OI.OP = OF.OC = (Prop. I. cor. 2. and V. 6. El.) OA^2, and, hence, by substitution, TL.TZ : TZ^2 : : OA^2 : OP^2. Again, the deflected ratio corresponding to the obliquity TZ or OP is evidently the ratio of OA to OP; whence TF : TZ : : OA : OP, and (V. 22. cor. 1.) TF^2 : TZ^2 : : OA^2 : OP^2. By identity of ratios, then, TF^2 : TZ^2 : : TL.TZ : TZ^2; consequently TF^2 = TL.TZ, and (V. 6. El.) TL : TF : : TF : TZ. Wherefore the triangles LTF and FTZ, having a common angle at T, are similar (VI. 13. El.), and hence TFZ being equal to TLF is a right angle; TZ is therefore a tangent.

Cor. 1. Hence, in the ellipse, if a diameter SO*s* be drawn parallel to a tangent TZ, applied at the extremity T of another diameter, the tangent SR applied to its own extremity S, will reciprocally be parallel to this other diameter TO*t*. For OFC and MFI being at right angles to the sides MP and OP of the triangle MOP, it follows from the sixth supplementary proposition to the Third Book of the Elements of Geometry, that PFK is perpendicular to OM; but it must be perpendicular also to the tangent SR, and consequently SR is parallel to TO*t*. Following out the principle of analogy, SO*s* (fig. 36.) parallel to the tangent at T, is the *secondary* or *imaginary* diameter in the hyperbola, and SR drawn parallel to OT is reciprocally a tangent to the derivative or interposed hyperbola at S.

Cor. 2. Hence, in the same curves, if the tangents MG and M*g*, applied at the extremities of a focal chord, be produced to meet the extension of the parallel diameter SO*s*, they will intercept a portion V*v*, which is equal to the transverse axis. For the chord G*g* being produced to meet the directrix in Q, will evidently be divided internally in F, and externally in Q in the same deflected ratio; whence, from the property of parallel and diverging lines, V*v* is similarly divided in I and P, or IV : I*v* : : PV: P*v*; and therefore (VI. 7. El.) OI : OV : : OV : OP, and (V. 6. El.) OV^2 = OI.OP. But the rectangle OI, OP was shown to be equivalent to the square of OA; which is consequently equivalent to that of OV, and thence the line OV itself is equal to OA.

Cor. 3. Hence, likewise, in the ellipse and hyperbola, the semitransverse axis OA is a mean proportional between the segment OK intercepted by a perpendicular from the focus, and the whole extension OM to the directrix. For the quadrilateral figure FCMK having right angles at C and F, is contained in a circle; wherefore the rectangle MO, OK, is equivalent to CO, OF, or to the square of OA, and hence OM : OA : : OA : OK. In like manner, it is shewn that OP: OA : : OA : OX.

Cor. 4. Hence, in those curves, any semidiameter OT is a mean proportional between the segment ON, intercepted by a focal ordinate and the extension OM to the directrix.

For (Prop. IX. cor. 2.) FM bisects the adjacent or vertical angle of the triangle TFt, and consequently FN, which is perpendicular to FM, bisects the interior or exterior angle; whence the base Tt is cut internally and externally in the same ratio, and therefore (VI. 7.) ON : OT : : OT : OM. Hence also in both curves, OX : OS : : OS : OP, the parallel OS representing a conjugate diameter in the hyperbola.

PROP. XI. THEOR.

A tangent to the curve makes equal angles in the parabola, with the straight lines drawn from the point of contact, one to the focus, and the other parallel to the axis; and in the ellipse and hyperbola, with straight lines drawn from that point to the two foci.

If, in the parabola, (fig. 26.) TM touch the curve, and TF be drawn to the focus, and TN parallel to the axis; TM will bisect the angle FTN. And, in the ellipse and hyperbola, (fig. 27. and 28.) if TF and Tf be drawn from the point of contact to the two foci, the tangent TM will bisect externally or internally the angle FTf contained by those lines.

1. *In the Parabola.*

For (fig. 26.) NLF being joined, since TL must be perpendicular to it (Prop. 10.), the right-angled triangles FLT and NLT, having also the side FT equal to NT, and TL common, are equal (I. 21. El.), and consequently the angle FTL is equal to NTL.

2. *In the Ellipse and Hyperbola.*

Draw the diameter TOt (fig. 27. and 28.), and from its remote extremity let tl be drawn parallel to TL, and join Ft. Because PF is perpendicular to TL and tl, the triangles FLT and Flt are right-angled, and consequently (I. 30. El.) the angles FTL and Ftl are complementary to TFL and tFl; but TFL and tFl are (Prop. V. cor. 3.) equal, and therefore FTL is equal to Ftl or (I. 22. El.) to fTm.

Cor. 1. Hence a straight line TQ drawn from the extremity of a diameter through the focus to meet another diameter parallel to the tangent, is equal to the semitransverse axis OA. For draw fv parallel to TQ, and meeting QO in v; and the triangles FOQ and fOv being evidently equal, FQ is equal to fv; but the triangles QTq and qfv are isosceles, since (I. 22. El.) the angles FQq and TqQ, or fqv and fvq, are equal to FTM and fTm; consequently TQ is equal to Tq, and fq to Fv. But FQ is equal to fv or fq; whence FT and fT, the one being greater than TF by FQ, and the other being less than fT by the same difference, are together the double of TQ; that is, (Prop. 6.) TQ is equal to OA.

Cor. 2. Hence in the parabola (fig. 26.), the angle which any two tangents TQ and PQ form at their point of concourse, is half of the angle TFP made on the same side by the lines drawn from the focus to the points of contact. For produce TQ to meet the extension of the axis in R. Since the angle FTR is equal to NTM, it is likewise equal to TRF. For the same reason, the angle FPS is equal to FSP; whence (I. 30. El.) the exterior angles TFZ and PFZ are the doubles of TRF and FSP or RSQ; and consequently the whole angle TFP is the double of TQP, the exterior angle of the triangle RQS.

Cor. 3. Hence also in the parabola (fig. 26.), the distance FT of the focus from any point of contact, is equal to its distance from R, the intersection of the tangent with the axis; and if a perpendicular TX be let fall, the base RX will be bisected by the vertex of the curve. For the triangle TFR is evidently isosceles, and FR being equal to FT, is equal likewise to NT or CX, and, taking away the equal portions AF and AC, there remains AR equal to AX.

PROP. XII. THEOR.

Any chord drawn parallel to a tangent, is bisected by the diameter which passes through the point of contact.

A diameter TH (fig. 84. 85. and 86.) bisects every chord DD′, which is parallel to the tangent applied at its vertex T.

1. *In the Parabola.*

Let (fig. 84.) the diameter and the chord be produced to meet the directrix in the points M and L; about the intersection H, with a radius equal to the apportionate distance, (which is in this case evidently HM,) describe a circle, join MF, and extend it through K to meet the circumference again in N, through the focus draw LG′FG, parallel to the chord, join HG and HG′, FD and FD′, and lastly HN.

It is obvious, from the construction employed in Proposition IX, that the focal lines FD and FD′ must be respectively parallel to G′H′ and GH. But, from Prop. V. cor. 3. MF is perpendicular to the tangent at T, and consequently cuts the parallel chord DHD′ at right angles; whence (I. 21. El.) the triangles MKH and NKH are equal, and therefore the angle MHK is equal to NHK. Consequently the triangles MHL and NHL (I. 3. El.) are equal, and both of them right angled; and hence LM and LN are two tangents to the circle. Wherefore, (VI. 9. El.) the line LG is divided harmonically in G′ and F by the circumference and the chord MN, or LG : GF : : LG′ : G′F. But, from the property of parallel and diverging lines, LG′ : G′F : : LH : HD, and LG : GF : : LH : HD′; whence, by identity of ratios, LH : HD : : LH : HD′, and, therefore, (V. 4. El.) HD is equal to HD′.

2. *In the Ellipse and Hyperbola.*

Produce (fig. 85. and 86.) OT and DD′ to meet the directrix in M and L, about H as a centre with the apportionate distance describe a circle, which will not reach the directrix in the ellipse, but will extend beyond it in the hyperbola, join MF, and produce it to terminate in the circumference at the points N and N′; draw OIP parallel to the chord, extend LG′ FG through the focus, and join LN and LN′, HN and HN′, HG and HG′, and FD and FD′.

From the property of parallel and diverging lines, OM : HM : : OP : HL : : OI : HK, and by composition $OM^2 : HM^2$: : OP.OI : HL.HK. But OM is to HM, as the apportionate distance of the centre O is to the apportionate distance of the point H, or as the semitransverse OA to the radius HN; and consequently $OA^2 : HN^2$: : OP.OI : HL.HK. Now, the rectangle OP, OI was proved to be equivalent to the square of OA, and therefore the rectangle HL, HK, must be equivalent to the square of HN. Whence (V. 6. El.) HL : HN : : HN : HK, and the triangles LNH and NKH or N′KH having likewise a common angle, are (VI. 13. El.) similar; consequently, since NK is perpendicular to the tangent of the curve at T or *t*, or to the parallel HL, the angles HNL and

HN′L are right angles, and hence LN and LN′ both touch the circle. Whence, as before, the oblique line LG is harmonically divided in the points F and G′. But FD and FD′ being parallel to HG′ and HG, therefore LG : GF : : LH : HD′, and LG′ : G′F : : LH : HD; whence HD is equal to HD′.

PROP. XIII. PROB.

Given the focus, the directrix, and the determining ratio, to draw a tangent parallel to a straight line given in position.

Let it be required to apply a tangent HT parallel to DE, (fig. 87. 88. and 89.).

Suppose, through the point of contact, a diameter to be drawn, meeting the directrix in M, and join this point with the focus. Since FM is perpendicular to the tangent, it must likewise be perpendicular to the parallel DE; consequently the point M, the diameter MT, and the point of contact T are all given in position.

Or, if any point within the curve were assumed, and the chord KL drawn, this would be bisected by the diameter MT; but the middle point I is evidently given, and therefore the point of contact T. Whence result two different ways of constructing the problem.

1. From the focus, let fall a perpendicular upon the given line, and produce it to meet the directrix in M; through M draw a diameter cutting the curve in T, and parallel to DE draw HT, which will be the tangent required.

2. Through any point assumed within the curve, draw the chord KL, which bisect in I, and through this extend a diameter to meet the curve in T, the point of contact.

In the parabola, the diameter being parallel to the axis will not meet the curve again; but in the ellipse and hyperbola, the diameter passing through the centre O will again meet the curve in another point of contact *t*.

PROP. XIV. PROB.

The focus, the directrix, and the determining ratio being given, to draw, from a given point, a tangent to the curve.

CASE. I. *When the given point is in the curve itself.*

Let it be required, from the point T, to draw a tangent to the curve.

1. *In the Parabola.* Suppose (fig. 26.) the tangent TM, and TN a parallel to the axis, to be produced to the directrix, and MF and NLF drawn to the focus. It follows, from Prop. IX. cor. 3. that MF is perpendicular to FT, from Prop. X. that NLF cuts TM at right angles, from Prop. XI. that TM bisects the angle FTN; and, from the third corollary to the same proposition, FR is equal to FT. Whence four different modes of construction :—1. Having joined FT, draw FM perpendicular to it, meeting the directrix in M, and join TM. 2. Having drawn TN parallel to the axis, join FN, and upon this let fall the perpendicular TLM. 3. Having joined FT, and drawn TN parallel to the axis, bisect the angle FTN by the line TM. 4. Produce the axis till FR be equal to FT, and TR being joined, is the tangent required. The demonstrations are obvious.

2. *In the Ellipse or Hyperbola.* Suppose (fig. 27. and 28.) the tangent TM and the line OT joining the centre,

to be produced to the directrix, and that FM and FP are joined, and TF and T*f* drawn to the two foci. Then, from the propositions already cited, MF is perpendicular to FT and PF to MT, and TM bisects externally (fig. 27.) or internally (fig. 28.) the angle FT*f*. Whence result three several ways of constructing the problem. 1. Having joined FT, draw FM perpendicular to it, meeting the directrix in M, and join TM. 2. Having found the centre O, draw OTP to the directrix, join FP, and let fall upon it the perpendicular TLM. 3. Having joined both foci with the point T, draw TM to bisect externally or internally the angle FT*f*.

Case II. *When the given point is in the directrix.*

1. *In the Parabola.* Suppose (fig. 26.) the tangent MT were drawn, and TN being let fall perpendicular upon the directrix, join FM and FN. It has been shown, that TF and TM are perpendicular to FM and FN; and it is obvious, that the right-angled triangle MFT is equal to MNT. Whence three different modes of construction. 1. Having joined MF, draw FT perpendicular to it, and join TM. 2. Make MN equal to MF, draw NT parallel to the axis, and join TM. 3. Draw MT bisecting the angle FMN.

2. *In the Ellipse and Hyperbola.* Since (fig. 27. and 28.) the tangent TM being produced to the directrix, and TM and FT joined, the angle MFT is a right angle; it is only required to draw FT perpendicular to FM, and cutting the curve in T and *t*, which are the points of contact.

Case III. *When the given point is in the axis.*

1. *In the Parabola.* Suppose (fig. 26.) a tangent RT

were drawn from a point R in the axis; join T with the focus, and draw TN parallel to the axis. Since (Prop. XI. cor. 3.) the triangle TFR is isosceles, the only construction required is to inflect FT equal to FR, and join TR.

2. *In the Ellipse and Hyperbola.* Since TR bisects externally (fig. 27.) or internally (fig. 28.) the angle FT*f*, a line TS perpendicular to it will bisect the same angle internally or externally; and, therefore, the angle RTS is a right angle, and (VI. 10. and 7. El.) OS : OF : : OF : OR. Whence this construction: Find OS a third proportional to OR and OF, and upon RS as a diameter describe a circle, cutting the curve in T and *t*, the points of contact.

Case IV. *When the given point lies any where without the curve.*

Let it be required from the exterior point P (fig. 29. 30. and 31.) to draw a tangent PT to the curve.

ANALYSIS.

Produce TP to the directrix; join FM, FP, and FT, and parallel to this, draw PQ. The triangles MTF and MPQ being evidently similar, PQ is to PM, as TF to TM, or in the deflected ratio corresponding to the position TM; whence PQ is the apportionate distance of the point P, and consequently given. But TF and PQ being perpendicular to MF, the triangle FQP is right angled, and it has the hypotenuse FP and the side PQ both given, and is therefore given in species and magnitude. Consequently the angle TFP is given, and FP given in position; the line MPT is hence given, and T the point of contact.

COMPOSITION.

From the point P (fig. 29. 30. and 31.) with a radius PQ equal to the apportionate distance, describe a circle, which will obviously touch the directrix in the parabola, (fig. 29.) lie within it in the ellipse, (fig. 30.) and extend beyond it in the hyperbola (fig. 31.); and from the focus draw FQ and F*q* (III. 22. El.) touching this circle and meeting the directrix in the points M and *m*; MPT and *mt*P, being joined, are the tangents required.

For join P, the centre of the circle, with its points of contact Q and *q*, and, parallel to these, draw FT and F*t*, meeting MP and *m*P, in T and *t*: From the property of parallel and diverging lines, FT is to TM, as PQ to PM, that is, by construction, in a deflected ratio corresponding to the inclination of PM; and F*t* is to *tm*, as P*q* to P*m*, which is likewise a given deflected ratio; consequently T and *t* are points in the curve. But the angles MFT and MF*t*, being equal (I. 22. El.) respectively to MQP and *mq*P, are (III. 19. El.) right angles, and therefore, by Prop. IX. cor. 3., MT and M*t* are tangents to the curve.

Scholium. It is obvious that this construction includes the two preceding cases. When the given point occurs in the directrix, the circle, with the terminations M and *m* of its tangents, all collaspe in the same point; and the contacts of the curve are therefore determined, as before, from the focal chord drawn at right angles to MF. When that given point is placed on the other side of the directrix, the circle described from it will in the parabola touch, and in the ellipse lie beyond, the directrix; in either case the points Q and *q* of circular contact occur beyond the points M and *m* of the directrix, and consequently both tangents may be always applied to the curve. But in the hyperbola, when the point P (fig. 32.) is situate beyond the centre, the points Q and q appear on the other side of M and m, and the tangents throw themselves to the opposite branch of the curve. If the given point falls in the centre O of the hyperbola (fig. 33.) the corresponding circle, being then described on the transverse axis Aq, will meet the directrix in the points of contact Q and q; for (Prop. 1. cor. 2.) OF, OA or OQ, and Oc being continued proportionals, the triangles OFQ and OQC, having likewise the angle at O common, are similar, and consequently the angle OQF is equal to the right angle OCQ, and (III. 24. El.) FQ touches the circle at Q. Wherefore the straight lines OQB$''$ and Oqb'' represent the tangents to the hyperbola, though they can never meet the curve, since the perpendiculars FT and Ft, which should mark the points of contact, are now parallel to them.

DEFINITION.

17. A straight line drawn on either side from the centre of an hyperbola in the extreme position of a vanishing tangent, is called an *Asymptote.*

Thus, in fig. 33., the straight lines OQ and Oq, determined by the intersection of circles described on Aa and OF, being the limits of all the tangents which can be applied to the curve, are *asymptotes* to the hyperbola.

PROP. XV. THEOR.

If a perpendicular to the principal axis of an hyperbola be extended to the asymptotes, the rectangle under its segments intercepted by the curve, is equivalent to the square of half the secondary axis.

Let the straight line IHI' (fig. 33.) cut the axis Aa at right angles; the rectangle ID, DI' is equivalent to the square of OB'.

For, from similar triangles, $OA : AB'' :: OH : IH$, and (V. 22. cor. 1. El.) $OA^2 : AB''^2 :: OH^2 : IH^2$, and by division, $OA^2 : AB''^2 :: OH^2 - OA^2 : IH^2 - AB''^2$. But the axis Aa being bisected in the centre O, $OH^2 - OA^2 =$ (II. 17. cor. 1. El.) $AH.Ha$; wherefore $OA^2 : AB''^2 :: AH.Ha : IH^2 - AB''^2$. Now, by Prop. 8. $OA^2 : AB''^2 :: AH.Ha : DH^2$, and consequently $IH^2 - AB''^2 = DH^2$ or $IH^2 = DH^2 + AB''^2$. But II' being evidently bisected in H, it follows that $IH^2 = DH^2 + ID.DI'$, and therefore $ID.DI'' = AB''^2$; but $AB'' = QF = OB'$, and $ID.DI' = OB'^2$.

Cor. 1. Hence, of a parallel to the axis, the rectangle under its segments intercepted by the asymptotes and the curve, is equivalent to the square of the semitransverse. For, from similar triangles $OA : AB'' :: Di' : DI :: Di : DI'$, and, by compounding the analogies, $OA^2 : AB''^2 :: Di.Di' : DI.DI'$; the second term being equal to the fourth, the first term must be equal to the third, or $OA^2 = Di.Di'$.

Cor. 2. Hence the asymptotes continually approach the curve, but never meet it. For let LKL' be another perpendicular to the axis more remote from the vertex A. Since $OB'^2 = ID.DI' = ML.LM'$, therefore (V. 6. El.) $ID : ML :: LM' : DI'$; but, from the divergency of the curve and its asymptotes, LM' is greater than DI', and consequently ID is greater than ML, or ML is less than ID.

PROP. XVI. THEOR.

Any straight line drawn through an hyperbola has equal segments intercepted on both sides, between the curve and its asymptotes.

The chord D*d* (fig. 36. and 37.) produced both ways to the asymptotes, has its exterior portion DY equal to *d y*.

For, having bisected D*d* in H, draw the diameter OH meeting the directrix in M, through which let WM*w* be drawn parallel to D*d*; join the focus F with the point B where the asymptote cuts the directrix; join also FW, F*w* and FM, meeting the chord D*d* in U, and at T apply the tangent TZ. By Prop. IX. cor. 3., TLF is a right angle, and consequently WM*w* is perpendicular to MF. But BF is perpendicular to OB, and OF bisects the angle BO*b*, and cuts B*b* at right angles; wherefore, by the seventh supplementary proposition to the Sixth Book of the Elements, the oblique line MW is equal to M*w*. Consequently, from the property of parallel and diverging lines, HY is likewise equal to H*y*; but, by the last proposition, HD is equal to H*d*, and therefore the remaining portion DY is equal to *dy*.

Cor. 1. Hence a tangent bounded by the asymptotes of an hyperbola is bisected in the point of contact. For (fig. 36.) the points of section, D and *d*, coalescing in T, it follows that TZ = T*z*; and, in the same manner, (fig. 37.) the points W and *w* coinciding T′, T′Z′ becomes equal to T′*z*′.

Cor. 2. Hence the same straight lines are likewise, on the other opposite sides, asymptotes to an interposed *secondary* or *conjugate* hyperbola, having for its axes those of the principal hyperbola interchanged. This secondary hyperbola is represented in figures 33. and 37. by two doted branches, the principal axis being B′b′.

Cor. 3. Hence, from the equality of the segments DY and *dy*, intercepted between the curve and its asymptotes, is derived an easy method of tracing an hyperbola from a series of points. Thus, in fig. 75., ON and OS being the asymptotes, and C a point in the curve; if, through C any straight line QCS be drawn, terminating in those asymptotes, and SR be made equal to QC, then will R be another point of the hyperbola.

DEFINITIONS.

18. A point which is bisected by any diameter is said to be *ordinately* applied, either half of it is called an *ordinate*, while the segments into which it divides the diameter are each termed an *absciss*.

19. In the parabola, the *parameter* of any diameter is equal to the double ordinate passing through the focus; but, in the ellipse and hyperbola, it is a third proportional to that diameter and the conjugate.

Thus, (fig. 43. 44. and 45.) DH*d* is *ordinately* applied to the diameter TO*t*, DH or *d*H is an *ordinate*, and the segments TH and H*t* are *abscisses*.

Again, in the parabola (fig. 34.), the chord GF*g* ordinately applied to the diameter TH is called the parameter of that diameter; and in the ellipse and hyperbola, (fig. 35. and 36.), the diameter T*t* is to its conjugate S*s*, as the same S*s* to the parameter of T*t*.

It follows that in the parabola, the parameter of a diameter is equal to four times the distance of its vertex from the focus. For GM and gM being joined, the right angled triangles MLT and FLT are evidently equal, and ML is equal to FL; and since TL and NF cut MF and MN proportionally, therefore NT is equal to TM or TF. But, by Prop. IX. cor. 4., the angle GMg is a right angle, and hence contained in a semicircle, of which the diameter is Gg and the centre N; consequently NG is equal to NM, and is the double of TM.

PROP. XVII. THEOR.

If a straight line, with a given inclination, meet a given line of the second order, the rectangle under its segments, intercepted from any point in it, has a given ratio to the difference of the squares of the real and apportionate distances of that point from the focus.

Let the straight line KDd (fig. 22. 23. 24. and 25.) making a given angle with the directrix or axis, meet the given curve in the points D and d; the rectangle under its segments DK, Kd has a given ratio to the difference between the square of FK, the real distance of any point K, from the focus, and the square of the apportionate distance of the same point.

For, produce KDd to meet the directrix in M; join FM, FD, and Fd, and draw KL and Kl respectively parallel to FD and Fd. It is evident, from Proposition VII, that the points L and l must lie in the circumference of a circle described from K, with a radius equal to its apportionate distance; and let FK be drawn to cut that circumference in the points I and i. From the property of parallel and diverging lines, KD : LF : : MK : ML, and

$Kd : lF :: MK : Ml$; wherefore, by compounding these analogies, $dK.KD : LF.Fl :: MK^2 : LM.Ml$. But (III. 26. El.) $LF.Fl = lF.Fi =$ (II. 17. El.) $FK^2 - KL^2$; and, for the same reason, $LM.Ml = vM.MV = MK^2 - KL^2$, or (fig. 25.) $KL^2 - MK^2$; whence $dK.KD : FK^2 - KL^2 :: MK^2 : MK^2 - KL^2$, or $KL^2 - MK^2$. Again, since KM has a given inclination, it has likewise a given ratio to the apportionate distance KL; and consequently the ratio of the square of MK to its difference from the square of KL is given. Wherefore the ratio of the rectangle under the intercepted segments KD and Kd to the difference of the squares of FK and KL, is given.

Cor. 1. Hence the rectangles under the intercepted segments of lines, drawn parallel from points either within or without the curve, are proportional. For K*a*A and CP*c* (fig. 40. 41. and 42.) being parallel, the rectangles AK*a* and CP*c* have each the same given ratio to the respective differences between the squares of the real and apportionate distances of the points K and P from the focus; and K*b*B and *d*PD being parallel, the rectangles BK*b* and DP*d* likewise have another given ratio to the same differences of these squares. Wherefore (V. 3. El.) $AKa : CPc :: BKb : DPd :: OG^2 : OH^2$.

Cor. 2. If the secant passes into a tangent, or the chord into a diameter, the rectangles under the segments become squares; and hence $A'K'^2 : B'K'^2 :: CPc : DPd$. For the same reason, tangents are proportional to their parallel diameters.

Cor. 3. Hence (fig. 52.) the chord E*e*, that joins the points of contact of two tangents, is ordinately applied to the diameter CA, which passes through T, their point of concourse. For draw OL and O*l* from the centre parallel to the tangents, and (Prop. XIII.) apply another tangent B*b* parallel to E*e*; join TA; and produce it to meet that chord in K. Because BE : *be* : : (VI. 1. El.) TE : T*e* : : OL : *ol*, and BE : BA : : OL : OG and *be* : *b*A : : O*l* : O*g*, it follows that BA : *b*A : : OG : O*g*, and therefore E*e* is bisected in K; but this chord (Prop. XII.) is likewise bisected by OA, which joins the point of contact A with the centre, and consequently TAK, with the diameter, forms one straight line.

PROP. XVIII. THEOR.

If, through any point in an hyperbola, a straight line be drawn with a given inclination and terminating in the asymptotes, the rectangle under its segments is equivalent to a given space.

If the straight line YD*y* (fig. 38.) cross the asymptotes at a given angle, and meet the curve at the point D or *d*, the rectangle YD, D*y*, or Y*d*, *dy*, is given; or, if D′Y*y*′*d*, with a given inclination, meet the opposite branches of the hyperbola, the rectangle D′Y, D′*y*′, or Y*d*′, *d*′*y*′, is likewise given.

For join the focus with the point Y in the asymptote, and the point B where that asymptote cuts the directrix and let fall the perpendicular YZ. Because YD*y* has a given inclination, the rectangle DY, Y*d* (by the last Proposition) bears a given ratio to the difference of the square of the real and of the apportionate distance of the point Y from the focus. But the right angled triangle YZB being evidently similar to OCB, YZ is to YB as OC to OB or OA, that is, in the determining ratio, and, consequently, by definition 12., YB is the apportionate focal distance of the point Y. Again, since the triangle FBY is right-angled, the square of FB is equivalent (II. 10. cor. El.) to the difference between the squares of YF and YB, or of the real and apportionate distances; but the perpendicular FB is given, and therefore the rectangle DY, Y*d*, which has a given ratio to it, is likewise given. Now (by Prop. XIII.), the segments DY and D*y* are equal to *dy* and *d*Y, and, consequently, the rectangle YD*y* is also given.

Cor. 1. Hence the rectangle under the segments DY and D*y* of any straight line drawn through a point of the curve and terminating in the asymptotes, is equivalent to the square of the intercepted parallel tangent QT, or of the parallel semidiameter. For *t*OTH being a diameter to D*d*, the tangent QT*q* is (Prop. XVI. cor. 1.) bisected at the point of contact T, and consequently the rectangle QT*q* is the same as the square of QT. Wherefore, since D*d* and QT, being parallel, have the same inclination to the asymptotes, the rectangle YD*y* and the square of QT bear likewise the same ratio to the square of FB, and are thence mutually equivalent. In the same manner, it is shown that the rectangle YD′*y*′ is equivalent to the rectangle TO*t*, or to the square of the semidiameter OT.

Cor. 2. Hence the rectangle under the straight lines D*y* and DX, drawn to the asymptotes at given angles from any point in the curve, is given; for the triangle XDY is evidently given in species, and consequently the ratio of DY to DX, and that of the rectangle DY, D*y*, to DX, D*y*, are given; but the rectangle DY, D*y* being given, the rectangle DX, D*y* is therefore likewise given. The gradual approximation of the asymptotes to the curve, without ever meeting it, is thence manifest.

Cor. 3. Hence rhomboids OADM and OBEN (fig. 61.) inscribed between an hyperbola and its asymptotes are equivalent; and consequently, (VI. 23. cor. 2. El.) straight lines AD and BE drawn from the curve parallel to one asymptote, are reciprocally as the distances OA and OB which they intercept on the other.

Hence also the triangles inscribed between the curve and its asymptotes are equivalent.

PROP. XIX. THEOR.

In the parabola, the square of an ordinate applied to any diameter, is equivalent to the rectangle under the absciss and the parameter of that diameter; but in the ellipse and hyperbola, the square of an ordinate is a fourth proportional to its diameter, the parameter, and the rectangle under the segments of that diameter.

1. *In the Parabola.*

Let TH (fig. 43.) be a diameter, to which DH*d* is ordinately applied; the rectangle under TH and its parameter is equivalent to the square of HD. For, draw the tangent TX, join FT and FM, which produce to meet D*d* in P, and join MI, M*i*. But the parallels IF*i* and DH*d* having the same inclination to the axis, and the apportionate distances of the points N and H from the focus being in this case equal to NM and HM; it follows, by Prop. XVII, that $NI^2 : HD^2 :: NM^2 - NF^2 : HM^2 - HF^2$. Now NF and HP being (Prop. VII. cor. 2.) perpendicular to MP, third term of the analogy $NM^2 - NF^2 =$ (I. 10. cor. El.) MF^2, and the fourth term $HM^2 - HF^2 =$ (II. 17. El.) $MF.2PX$ or $4MX.PX$; whence $NI^2 : HD^2 :: MF^2$ or $4MX.MX : 4MX.PX$ :: (V. 25. cor. 2. El.) $MX.PX$ :: (VI. 1. El.) $MT : TH$. But $NI^2 = NM^2 = 4MT.MT$, and therefore (V. 4. El.) $HD^2 = 4MT.TH$, or the square of HD is equivalent to the rectangle under TH and the parameter.

2. *In the Ellipse.*

Let DHd (fig. 44.) be ordinately applied to the diameter TOt, of which SOs is conjugate; then Tt^2 : Ss^2 : : THt : HD^2.

For Ss being parallel to Dd, the rectangles formed by the intersections at the points O and H are (Prop. XIV. cor. 1.) proportional, that is, TOt or TO^2 : SOs or SO^2 : : THt : DHd or HD^2, and consequently Tt^2 : Ss^2 : : THt : HD^2, or Tt : parameter : : THt : HD^2.

3. *In the Hyperbola.*

Let DHd (fig. 45.) be ordinately applied to the diameter toT, and SOS, equal to the tangent QTq, intercepted by the asymptotes, represents the conjugate, or is the imaginary or secondary diameter; then will Tt^2 : Ss^2 : : THt : HD^2.

For the tangent at T being parallel to the ordinate HD, it follows, that OT : TQ : : OH : HY, or (V. 22. cor. 1. El.) OT^2 : TQ^2 : : OH^2 : HY^2; and consequently (V. 19. cor. 1. El.) OT^2 : TQ^2 or OS^2 : : OH^2—OT^2 : HY^2—TQ^2. But (II. 17. El.) OH^2—OT^2 = THt, and, by cor. 1. Prop. XVIII. TQ^2 = YDy, or HY^2—TQ^2 = HY^2—YDy =(II. 17. cor. 1. El.) HD^2; whence OT^2 : TQ^2 : : THt : HD^2, and, therefore Tt^2 : Ss^2 : : THt : HD^2, or Tt : parameter : : THt : HD^2.

Cor. 1. Hence in the parabola (fig. 43.) the square of the perpendicular DK to a diameter, is equivalent to the rectangle under the absciss TH and the parameter of the axis.

Cor. 2. Hence also in the parabola, the rectangle under the segments DP′ and P′d of a chord is equivalent to the rectangle under the intercepted portion AP′ of any diameter cutting it and the parameter of the diameter TH to which the chord is ordinately applied. For, having drawn AV an ordinate or parallel to Dd, then $HD^2 = TH.4FT$, and VA^2 or $HP'^2 = TV.4FT$, and consequently $HD^2 - HP'^2 = DP'd = AP'.4FT$.

PROP. XX. THEOR.

Any straight line extended from the concourse of two tangents is divided harmonically by the curve and the chord which joins the points of contact.

Let CA and CB (fig. 46. 47. and 48.) be two tangents, and CDEG a straight line drawn through their point of concourse C to cut the curve at D, and terminate in remoter flexure at G, while at E it crosses the chord AB joining the points of contact; this line CG is harmonically divided in the intermediate points D and E, or CG : CD : : EG : ED.

For, through D and G, the points of section by the curve, draw the parallels IDL and KGM meeting the tangents in I, L, and K, M, and, through the point of concourse C, draw a diameter CD′G′. This diameter must (Prop. XVII. cor. 3.) bisect the parallel chords DN, AB, and GH; but (VI. 2. El.) it bisects likewise the transverse lines IL and KM; whence ID=NL, and KG=HM, or IN=DL and KH=GM. Again, from the property of parallel and diverging lines, CK : CI : : KG : ID : : GM : DL, and consequently, by composition of ratios, $CK^2 : CI^2$: : KGM or GKH : IDL or DIN. Now, since the secants KGH and IDN are parallel, $AK^2 : AI^2$: : GKH : DIN (Prop. XVII. cor. 2.); whence, by identity of ratios, $CK^2 : CI^2$: : $AK^2 : AI^2$, and therefore CK : CI : : AK : AI. But (VI. 1. El.) CK : CI : : CG : CD, and AK : AI : : EG : ED, and hence CG : CD : : EG : ED.

Cor. 1. Hence the intercepted portion DG of the secant is cut internally and externally at the points E and C in the same ratio; and consequently (VI. 7. El.) when this secant passes through the centre, the semidiameter OD′ is a mean proportional between the distances OE′ and OC intercepted by the ordinate AE′ and the tangent AC.

Cor. 2. Hence, if from any point B (fig. 49.) in the diameter of an ellipse, an ordinate AB be drawn, and a perpendicular BE erected to meet a circle described on CD, the tangents AT and ET applied at their extremities will concur in the same point T of the diameter produced. For join OE, and from the similar triangles OBE and OET, OB : OE or OC : : OE or OC : OT; the same property as in the ellipse, and consequently the point T is identical.

Cor. 3. When the straight line CDE or C*de* drawn through the point of concourse C or C′, to meet the curve and the chord joining the points of contact, is parallel to the axis in the parabola, or when it is (fig. 47.) parallel to either asymptote in the hyperbola, it will evidently never meet the curve again, and must, therefore, have its exterior and interior segments C*d* and *de* equal. In the parabola this property is easily demonstrated directly; for (fig. 48.) draw *dm* parallel to CA, meeting the diameter from A, upon which let fall, from A and B, the perpendiculars A*l* and B*k*. By cor. 1. to the last proposition, the squares of these lines are equivalent respectively to the rectangles under the abscisses A*m* and *de*, and the parameter of the axis; wherefore A*m* or C*d* is equal to *de*.

Cor. 4. Hence if two tangents AD and BD to a parabola (fig. 94.), be cut by a third tangent ECG, their segments will be mutually proportional. For the diameters DH, EI and GL will bisect the chords AB, AC and BC; whence AH=BH, AI=IK, and KL=LB, and consequently IH = KL. Wherefore AE : ED :: AI : IH :: IK : KL :: EC : CG :: HL : LB :: DG : GB.

PROP. XXI. THEOR.

Chords drawn through the focus are proportional to the rectangles under their segments.

In fig. 17, 18, 19. and 21, the chord DD′ is to HH, as the rectangles under the segments DF and FD′, intercepted by the focus, to the rectangle under HF and FH′.

For draw the perpendicular ordinate GG′, bisect the chord DD′ in P, and produce it to meet the directrix in M. From the property of the curve, FG : FD :: FC : DE :: FM : DM; and for the same reason, FG′ : FD′ :: FM : D′M; whence FG or FG′ : 2PD :: FM : 2PM, or FG : PD :: FM : PM. But since by Prop. XVII., D′M is cut harmonically in the points F and D, it follows (VI. 7.) that PF : PD :: PD : PM, and therefore PF : PM :: PF^2 : PD^2; and, by division, FM : PM :: PD^2—PF^2 or DFD′ : PD^2. Whence, by identity of ratios, DFD′ : PD^2 :: FG : PD :: (V. 24. El.) FG.PD : PD^2; and therefore the rectangle under DF, FD′ is equivalent to the rectangle under PD and FG, or under the chord DD′, and the half of FG. In like manner, it may be shewn that the rectangle under FH and FH′ is equivalent to the rectangle under the chord HH′ and the half of FG. Wherefore the rectangles DFD′ and HFH′ are as the chords DD′ and HH′.

Cor. 1. Hence, in the ellipse and hyperbola, the rectangle under the transverse axis A*a*, and any focal chord DD′ is equivalent to the square of the parallel diameter IOI′. For, since the transverse axis itself is a focal chord,

$Aa : DD' :: AFa : DFD'$; but, by Prop. XVII. cor. 1., the rectangle AFa is to DFD' as the square of OA is to that of OI, or as the square of Aa to that of II'. Wherefore $Aa^2 : II'^2 :: Aa : DD' :: Aa^2 : Aa.DD$, and consequently $Aa.DD' = II'^2$.

Cor. 2. Hence in the ellipse and hyperbola (fig. 35. and 36.) a diameter Tt is to the transverse axis Aa, as the focal chord Gg to the parameter of that diameter to which it is ordinately applied. For $Tt : Ss :: Ss :$ parameter, or $Ss^2 = Tt$. parameter; but $Ss^2 = Aa.Gg$, and consequently Tt. parameter $= Aa.Gg$, or $Tt : Aa :: Gg :$ parameter of Tt.

PROP. XXII. THEOR.

The sum of the squares of two conjugate diameters in the same ellipse, and their difference in the same hyperbola, is equivalent to a given space.

1. *In the Ellipse.*

Let SOs and TOt (fig. 90.) be two conjugate diameters; their squares are together equivalent to the squares of the two axes.

For produce the conjugate diameters to meet the directrix in M and P, join MF and PF, and extend them to K and I; through the focus draw FN and FX parallel to PO and MO, and upon FO let fall the perpendiculars NY and ZX.

Since (Prop. X. cor. 4.) $OM.ON = OT^2$, and (Prop. X. cor. 3.) $OM.OK = OA^2 = COF$, therefore $OT^2 = COF - OM.KN$. But, FN being parallel to PO, the angle KNF is equal to MOI, and the angles at K and at I are both right angles; whence the triangle KFN is similar to IMO, and consequently $KN : FN :: OI : OM$, and $OM.KN = OI.FN$ or IOX. Wherefore, by substitution, $OT^2 = COF - IOX$; and in the same manner, it is proved that $OS^2 = COF - KON$. Now, the right-angled triangles OIF and OZX being evidently similar, $OF : OI :: OX : OZ$, and $OF.OZ = OI.OX$; whence $OT^2 = COF - ZOF = OF.CZ$. For the same reason, $OS^2 = OF.CY$, and consequently $OT^2 + OS^2 = OF\ (CZ + CY)OF.CF + OF\ (CZ + FY)$. Now, $OF.CF = OB^2$, and the triangles OYN and FZX being evidently equal, the segments FZ and FY are together equal to the whole diagonal OF, and hence $OF\ (CZ + FY) = OF.OC = OA^2$. Wherefore $Tt^2 + Ss = Aa^2 + Bb^2$.

2. *In the Hyperbola.*

Let T (fig. 91.) be any point in the curve, from which OT is drawn to the centre, and the tangent G, TE to the asymptotes; GE was shown (XVIII. cor. 1.) to be equal to the imaginary or secondary conjugate diameter, and the difference between its square and that of twice OC remains the same in every position.

For the same construction being employed as before, only changing the position of the lines, it is obvious that $OT^2 = OM.ON = OM.OK + OM.KN$. But the right-angled triangles OCM and OIM are similar to OKF and NKF, and therefore $OM : OC :: OF : OK$, and $OM : OI :: FN : KN$; consequently $OM.OK = OF.OC$, and $OM.KN = OI.FN$. But the right-angled triangles OIF and FYN being similar, $OI : OF :: FY : FN$, and $OI.FN = OF.FY = OF.OZ$. Whence, by substitution, $OT^2 = OF.OC + OF.OZ = OF.CZ$. In like manner, it may be proved that $OS^2 = OF.CY$, wherefore $OT^2 - OS^2 = OF(CZ - CY) = OF(OC - CF)^2 = OF.OC - OF.CF =$

$OA^2—OB^2$; and consequently $OT^2—OS^2 = OA^2—OB'^2$, and thence $Tt^2—Ss^2 = Aa^2—B'b^2$.

Cor. 1. Hence, in an equilateral hyperbola which corresponds to the circle, since the axes are equal, any conjugate diameters must also be constantly equal. This appears likewise from the inspection of fig. 58.; for the triangle QO*q* being right-angled, the middle point T of Q*q* is the centre of a semicircle described on that base, and consequently the radius TO = TQ; but TQ is equal to the semidiameter which is secondary or conjugate to OT.

Cor. 2. Hence in the ellipse, the squares of the sum and of the difference of two conjugate diameters are together constant in the same curve. For, it is evident from II. 18. El., that the squares of the sum and of the difference of any two lines are equivalent to twice their squares.

PROP. XXIII. THEOR.

If two tangents applied at the extremities of a diameter meet an oblique tangent, their rectangle is equivalent to the square of a parallel semidiameter; and the rectangle under the segments of this third tangent at its point of contact, is equivalent to the square of a semidiameter drawn parallel to it.

Let (fig. 52. and 55.) AB and CD be two tangents applied at the extremities of the diameter AC, and limited by a third tangent BED; and, if OG and OL be drawn from the centre parallel to AB or CD and BD, then will $AB.CD = OG^2$ and $BE.ED = OL^2$.

For produce the diameter AC and the tangent BD to meet in T; also produce OG to H, and draw EI, EK parallel to AC and AB. From the property of parallel and diverging lines, AT : AB : : TK : KE : : TO : OH : : TC : CD. And consequently (V. 22. El.) $AT^2 : AB^2 :: AT.TC : AB.CD :: KT.TO : KE.OH$; but (Prop. XX. cor. 2.) AT.TC = KT.TO, and therefore AB.CD = KE.OH or OI.OH. Now (Prop. XX.) OI, OG and OH being continued proportionals $OI.OH = OG^2$, and hence $AB.CD = OG^2$. Again, because the tangents CD and ED are parallel to the semidiameters OG and OL, therefore (Prop. XVII. cor. 2.) $CD^2 : ED^2 :: OG^2 : OL^2$; but the lines TC and TD being divided similarly by the several parallels, and OK, OA and OT being proportionals, it follows that HE, HB and HT are likewise in continued proportion; and hence AB : CD : : BT : TD : : BE : ED, and (V. 25. cor. 2.) $AB.CD : CD^2 :: BE.ED : ED^2$: or alternately $AB.CD : BE.ED :: CD^2 : ED^2$. Wherefore $AB.CD : BE.ED :: OG^2 : OL^2$, and since it was proved that $AB.CD = OG^2$, therefore $BE.ED = OL^2$.

Cor. Hence $TE.EH = OL^2$. For TD : ED : : TB : BE, whence TD : ED : : DB or 2BH : 2EH : : BH : EH, or by division, TE : ED : : BE : EH, and therefore BE.ED or $OL^2 = TE.EH$.

PROP. XXIV. THEOR.

The rhomboid which circumscribes a given line of the second order, contains a given space.

Let DE and GH (fig. 54. and 57.) be conjugate diameters, and KI, LM, and KL, IM tangents applied at their extremities, and which will evidently form a parallelogram; this figure IKLM is equivalent to the rectangle under the two axes A*a* and B*b*.

For, in figures 35. and 36., from the properties already demonstrated, and from those of parallel and diverging lines, it is evident that MO : MN : : MI : MF : : OI : OX : : OI.OP or OA^2 : OX.OP or OS^2 ; and again MO : MN : : MI : MF : : MI.IF or IL^2 : MF.IF or OF.FC, that is, OB^2. Wherefore, by identity of ratios, OA^2 : OS^2 : : IL^2 : OB^2, and consequently OA : OS : : IL : OB; whence the rectangle under the semiaxes OA and OB is equivalent to the rectangle under the semidiameter OS and the perpendicular IL, which is obviously the area of the rhomboid TS. Whence the whole circumscribing rhomboid, being the quadruple of this, must contain a space equivalent to the rectangle under both the axes A*a* and B*b*.

Cor. Hence the inscribed rhomboid formed by joining the extremities of the two axes is equal to half their rectangles.

PROP. XXV. THEOR.

In the ellipse and hyperbola, the rectangle under perpendiculars let fall from the foci upon any tangent to the curve, is equivalent to the square of the semiconjugate axis; and the rectangle under straight lines which join the foci with the point of contact, is equivalent to the square of a semidiameter drawn parallel to the tangent.

Let (fig. 53. and 56.) DCE be a tangent to the curve, FD and *f*E perpendiculars let fall upon it from the two foci, FC and *f*C straight lines drawn likewise from them to the point of contact, OI a semidiameter parallel to DE, and OB the semiconjugate axis; then will $FD.fE = OB^2$, and $FC.fC = OI^2$.

For join OD, OC, OE, and produce FD to meet fC and OE in the points G and H. Because the exterior angle FCG is bisected by the tangent, the right angled triangles CDF and CDG, having the angle $FCD = GCD$, and the side CD common, are equal; wherefore $GC = FC$, and $fC \pm GC = fC \pm FC$. But the transverse axis Aa is equal to $fC + FC$ in the ellipse, and to $fC - FC$ in the hyperbola; whence OD, as it bisects FG and Ff, is equal to the half of Aa or to OA, and consequently the point D lies in the circumference of a circle described upon the transverse axis. In like manner, it is shewn that the point E lies in the same circle. Next, the triangles OHF and OEf, having their vertical angles at O equal, the angle OFH equal to the alternate angle OfE, and the side OF equal to Of, are equal, (I. 21. El.), and therefore OH is equal to OE, and the point H lies in the same circumference. But from the property of the circle, the rectangle under FD and FH or fE is equivalent to the rectangle under AF and Fa, that is, (Prop. 6. cor. 3.) to the square OB. Again, since $Aa = fC \pm FC$, therefore (II. 15. and 16. El.) $Aa^2 = fC^2 + FC^2 \pm 2FC.fC$; but (II. 22. El.) $fC^2 + FC^2 = 2OC^2 + 2OF^2$, and consequently $2OA^2 = OC^2 + OF^2 \pm FC.fC$. Now (Prop. 22.) $OA^2 \pm OB^2 = OC^2 \pm OI^2$, and adding OF^2 to both, $2OA^2 = OC^2 + OF^2 \pm OI^2$, whence $FC.fC = OI^2$.

Cor. 1. Hence if a perpendicular ON (fig. 53. and 56.) be let fall from the centre upon a tangent, $OA : ON :: OI : OB$. For from similar triangles, $FC : FD :: fC : fE$, and (V. 19. El.) $FC : FD :: fC \pm FC$ or $2OA : fE \pm FD$ or $2ON$; but by composition of ratios, $FC^2 : FD^2 : FC.fC$ or $OI^2 : FD.fE$ or OB^2, and consequently $OA^2 : ON^2 :: OI^2 : OB^2$, or $OA : ON :: OI : OB$.

Cor. 2. Hence also $FC : FD :: OI : OB$. For, by the preceding corollary, $FC^2 : FD^2 :: FC.fC$ or $OI^2 : FD.fE$, or OB^2.

Cor. 3. Hence in the ellipse (fig. 53.) if a perpendicular CR, at the contact of a tangent, and which is sometimes called a *normal*, be produced to meet the conjugate axis in r; then will $OA^2 : OB^2 :: Cr : CR$. For apply the ordinate CP to the transverse axis, and since CR evidently bisects the angle FCf, it follows that FC is to FR as fC to fR, or as $FC+fC$ or Aa to $FR+fR$ or Ff; whence $OA^2 : OF^2 : fC^2-FC^2$, or $fP^2-FP^2 : fR^2-FR^2$, or $OA^2 : OF^2 :: OF.OP : OF.OR :: OP : OR$; but $OP : OR :: Cr : rR$, and consequently $OA^2 : OF^2 :: Cr : rR$, or $OA^2 : OB^2 :: Cr : RC$.

PROP. XXVI. THEOR.

In the ellipse the rectangle under a straight line drawn from the focus to the curve, and the semitransverse axis increased or diminished by the internal or external portion intercepted between the focus and a perpendicular let fall upon it from the centre, is equivalent to the square of the semiconjugate.

Upon a straight line FD or Fd (fig. 62.) drawn from the focus to the curve, let fall the perpendicular OH from the centre intercepting a segment FH, then, Aa and Bb being the two axes, $FD\,(OA + FH) = OB^2$ or $Fd\,(OA-FH) = OB^2$.

For let CE be the directrix, produce the axis Aa to meet it, and draw DE or de parallel to OC, and DG or dg parallel to CE. The right-angled triangles DFG, or dFg, and OFH being evidently similar, $FD : FG$, or $Fd : Fg :: OF : FH$, and hence (V. 6. El.) $FD.FH =$

$OF.FG$, or $Fd.FH = Fg.OF$. But, from the nature of the curve, $FD : DE$ or CG, or $Fd : Cg :: OF : OA$, and consequently $FD.OA = OF.CG$, or $Fd.OA = Cg.OF$. Wherefore, by combining these equivalent rectangles, $FD.OA + FD.FH = OF.Cg + OF.FG$, and $Fd.OA - FD.FH = OF.Cg - OF.Fg$, that is, $FD(OA + FH) = OF(CG + FG) = OF.CF$, or $Fd(OA - FH) = OF(Cg - Fg) = OF.CF$. Now (Prop. VI. cor. 3. $OF.CF = AF.Fa = OB^2$, and hence $FD(OA + FH) = OB^2 = Fd(OA - FH)$.

Cor. 1. It is evident that the same property might be extended to the hyperbola. But in this case the rectangle under FD, and the excess of OA above FH, is equivalent to the square of the secondary semiaxis, or to the excess of the eccentricity above that of the semitransverse. As the line from the focus approaches to a parallelism with the asymptote, FH becomes nearer equal to OA, and consequently FD tends towards indefinite extension.

Cor. 2. Hence in the parabola (fig. 66.) if, from the point C in the directrix, a perpendicular CI be let fall upon FD; then will $FD(CF + FI) = CF^2$, or $Fd(CF - FI) = CF^2$. For the triangles CFI and DFG, or dFg, being similar, $CF : FI :: FD : FG$, or $Fd : Fg$, and therefore $CF.FG = FD.FI$, or $CG.FI$, and $CF.Fg = Fd.FI$, or $Cg.FI$; with each of these conjoin $CF.CG$ or $CF.Cg$, and $CF^2 = FD(CF + FI)$, or $CF^2 = Fd(CF - FI)$.

Scholium. This proposition is of singular use in astronomy. The semitransverse axis OA (fig. 62.) being expressed by unit, let the semiconjugate be denoted by b, the eccentricity OF by ε, the *radius vector* FD by r, and the *true anomaly*, or the angle AFD, by ϕ; then $FH = \varepsilon \cos \phi$, and $b^2 = 1 - \varepsilon^2 = r(1 + \varepsilon \cos \phi)$, and consequently $r = \dfrac{1 - \varepsilon^2}{1 + \varepsilon \cos \phi}$.

For the cometary motions, the parabola may be assumed with sufficient accuracy. In this case, let the parameter of the axis be denominated by p, and $\frac{p^2}{4} = r\left(\frac{p}{2} + \frac{p}{2} \cos\phi\right)$, or $p^2 = r\,(2p + 2p \cos \phi)$; whence $r = \frac{p}{2}\left(\frac{1}{1 + \cos \phi}\right)$.

These expressions are easily expanded into infinite series, in which again the powers of $\cos \phi$ may, by the application of the arithmetic of sines, be changed into the cosines or sines of multiple arcs.

PROP. XXVII. THEOR.

If one of the asymptotes of an hyperbola be divided into segments in continued proportion, by parallels to the other asymptote, straight lines drawn from the centre to the point where these meet the curve, will contain equal sectors.

In the asymptote OI (fig. 61.) of an hyperbola, let distances from the centre be assumed, such that OA is to OB, as OB to OC; and, from the points of division, draw AD, BE and CF parallel to the other asymptote OL, and meeting the curve in D, E and F; then if, from the centre, the radiating lines OD, OE, and OF be drawn, they will form the sector DOE equivalent to EOF.

For, at the middle point E, apply a tangent to the curve, and join the points D and F by a chord, extending both these lines to meet the asymptotes in H,K and I,L; and produce the radiating line OE to meet the chord DF in G. From the property of the asymptotes, the intercepted portion EK is equal to EH, and DL to FI; whence

(VI. 1. El.) the corresponding segment OB is equal to BH, and OA to CI. Again, the parallels AD, BE, and CF are (Prop. XVIII. cor. 3.) reciprocally as the distances OA, OB and OC from the centre, and consequently CF is to BE as OB to OC, or as OA to OB; that is, CF is to BE, as CI to BH, or alternately CF is to CI, as BE to BH; wherefore the triangles EBH and FCI, having the sides about the equal angles at B and C thus proportional, are similar; and hence the angle BHE is equal to CIF, and the tangent HEK is (I. 22. El.) parallel to IGL. Consequently, the diameter OE bisects the chord DF, and therefore the curve space DEFD, which may be made up of these chords; taking away, therefore, the equal spaces EDG and EFG from the equal (II. 2. El.) triangles ODG and OFG, there remains the sector DOE equal to EOF.

Cor. 1. Hence one of the asymptotes of an hyperbola being divided into continued proportion, straight lines drawn from the points of section parallel to the other asymptote, will contain equal spaces without the curve. For since (Prop. XVIII. cor. 3.) the rhomboid OADM is equivalent to OBEN, and they are bisected by their diagonals OD and OE, it is evident, that the two triangles ODM and OEB are together equivalent to OADM, consequently taking these severally away from the mixtilineal space OBEDM, there remains the sector ODE equivalent to the space ABED. In the same manner, it is shown, that the sector OEF is equivalent to BCFE, and, consequently, the successive spaces ABED and BCFE are equivalent.

Cor. 2. Hence, from the nature of logarithms, the sectors ODE, ODF, &c. or the collective spaces ADEB, ADEFC, &c. will represent the logarithms of the ratios of OA to OB, to OC, &c.; and, if OA denote the unit of the arithmetical scale, the sector ODE, or the space ADEB, will express the logarithm of OB on any logarithmic system depending on the angle of the asymptotes.

DEFINITION.

20. A circle is said to *osculate* a curve, when no other circle can be made to pass between it and the curve.

This species of contact of a circle with a curve is evidently closer than that of a mere straight line touching it. As the *tangent* is the limit of a secant, or the position to which a straight line, joining *two* adjacent points in the curve, continually approximates as these are brought nearer; so the *osculating* or *equicurve circle* is the limit to which a circle, described through *three* adjacent points in the curve, continually approaches as their mutual interval contracts. While the tangent thus indicates the final direction of *two* contiguous points, the circle marks the ultimate arrangement of *three* points which are about to coalesce. The osculating circle may therefore be derived, either from the consideration of three approximating points, or from that of a tangent combined with a point merging the same contact.

PROP. XXVIII. THEOR.

An equicurve circle intercepts from a diameter passing through the point of osculation, a segment equal to the parameter of that diameter.

Let (fig. 65. 63. and 64.) a circle osculate the curve at the point C; it will cut off from the diameter drawn thro' that point, a chord CE, which is equal to the parameter of that diameter.

For assume a point *p* near the vertex of the diameter C*c*, to which apply the double ordinate *mpn*, and through the three points *m*, C and *n* describe a circle cutting C*c* in *e*. In the point, the ellipse and hyperbola (fig. 65. and 63.) by Prop. XI. the diameter C*c* is to its parameter CE, as the rectangle C*p*.*pc* to the rectangle *mp*.*pn*. But, from the property of the circle, the rectangle *mp*.*pn* is equivalent to the rectangle C*p*.*pe*. Wherefore C*e* is to CE, as the rectangle C*p*.*pc* to the rectangle C*p*.*pe*, or (V. 25. cor. 2.) as *pc* to *pe*, and alternately C*c* is to *pc*, as CE to *pe*; but the ratio of C*c* to *pc* approaches to equality, the nearer the point *p* is taken to C, and therefore the ratio CE to *pe* approximates continually to that of equality; or the point E is the ultimate limit of *e*, and the parameter CE is consequently the intercepted chord of the osculating circle.

In the parabola (fig. 64.) the rectangle under absciss C*p* and the parameter CE is equivalent to the square of *mp* or to the rectangle under *mp* and *pn*, that is, from the property of the circle, the rectangle under C*p* and *pe*; consequently *pe* is equal to CE, or the portion of the diameter intercepted by the equicurve circle is ultimately equal to its parameter.

Cor. 1. Hence in the ellipse and hyperbola, the square of a semiconjugate OD to the diameter CO*c* passing through the point of osculation, is equivalent to the rectangle under a perpendicular CK, let fall from that point on the conjugate and CH the radius of curvature.

For having drawn the diameter CHI of the equicurve circle, from its centre H let fall the perpendicular HG upon the chord CE; and, since this circle and the curve have the same tangent at C, it is evident that CHI must be perpendicular to that tangent or to the parallel diameter DO*d*, and that the perpendicular HG will bisect the chord CE in the point G. Now, the right-angled triangles CGH and CKO being evidently similar, CG : CH : : CK : CO, and since C*c* : D*d* : : D*d* : CE or CO : OD : : OD : CG, so CG.CO or OD^2 = CK.CH.

Cor. 2. Hence in the ellipse and hyperbola, (fig. 65. and 63.) the semiconjugate OD of the diameter passing through the point of osculation, is to the radius CH of curvature, as the rectangle under the semiaxes OA and OB, to the square of that semiconjugate OD. For (V. 25. cor. 2.) OD : CH : : OD.CK : CH.CK; but (Prop. XXV.) OD.CK = OA.OB, and by the last corollary, CH.CK = OD^2; whence OD : CH : : OA.OB : OD^2.

Cor. 3. In the same ellipse or hyperbola, the radius of curvature is proportional to the third power of the diameter which is conjugate to the one passing through the point of osculation. For the several lines in the last analogy being expressed numerically, it follows (V. 6. El.) that CH (OA.OB) = OD^3; but the axes being constant, CH must be proportional to OD^3 or Dd^3.

Cor. 4. Hence in the parabola (fig. 64.) the parameter of the axis is to the normal CP at the point of osculation, as the parameter of the diameter at that point to the radius of curvature. For the right-angled triangles CLP and CGH being obviously similar, PL : CP : : CG : CH, or 2PL : CP : : 2CG or CE : CH; but CE is the parameter of the diameter passing through C, and since (VI. 15. cor. El.) CL^2 = TL.LP or 2AL.LP, it is evident that 2PL is the parameter of the axis.

PROP. XXIX. PROB.

Given a portion of a Line of the Second Order, to find the axis and foci.

Let *k*AP (fig. 67.) be the portion of a curve, whose axis and foci are to be determined.

Assume in the curve any two points C and D, and draw the parallel chords C*c* and D*d*, which bisect in I and H by the straight line K*k*. Again, take other two points E and Gg, and draw the parallel chords E*e* and G*g*, which bisect likewise by the straight line N*n*. These straight lines K*k* and N*n* must evidently be diameters, and they will run parallel to each other in the case of the parabola, will converge in that of the ellipse, and diverge in that of the hyperbola. In these two last cases, the intersection of the diameters, K*k* and N*n*, will assign the centre C of the curve. Next, take any point P in the curve, and from the centre, with the radius OP, describe a circle meeting the curve again in *p*, join the chord P*p*, and bisect it at right angles by A*a*, which is obviously the transverse axis, B*b* perpendicular to it being the conjugate, and the foci F and *f* being determined by the intersection of a circle described from B with the radius OA. But in the hyperbola, the imaginary or secondary axis B*b'* is found from the analogy $AQ.Qa : QP^2 :: Aa^2 : Bb'^2$, and the foci found by making OF or O*f* equal to the hypotenuse AB.

In the case of the parabola (fig. 68.) assuming any point R in the curve, let fall on the diameter KIH, a perpendicular RX, and bisect it at right angles by the straight line AV, which is consequently the axis; join the vertex A with the point R, and draw the perpendicular RS, meeting the axis in S; and because ARS is right-angled, $RV^2 = AV.VS$; but (Prop. VIII.) $RV^2 = AV.$parameter, so the segment VS is equal to the parameter of the axis, and the fourth part of it therefore gives the focal distance AF.

PROP. XXX. PROB.

Given two conjugate diameters of an ellipse or hyperbola to determine the curve.

Let OC and CD (fig. 69. and 70.) be two semidiameters which are conjugate; it is required to find the axes and foci.

ANALYSIS.

Let A*a* and B*b* be the two axes required, and at C, the vertex of the semidiameter OC, apply a tangent, and produce it to meet those axes in the points I and K; on IK describe a circle, which (III. 22. El.) will pass through O, and again meet OC in E, and bisect OE by the perpendicular HG. From the property of the circle, $OC.CE = IC.CK$; but the tangent to the curve at C has (Prop. XIX, cor.) the rectangle under its segments IC, CK, intercepted by the axes, equivalent to the square of the parallel diameter OD, and consequently $OC.CE = OD^2$, and CE is thus a third proportional to the semidiameters OC and OD, or is equal to half the parameter of the diameter CO*c*. Wherefore the point E is given, and consequently the perpendicular GH which bisects OE and its concourse H with the tangent, are given; the circle described from this point H as a centre, is therefore given, and also its intersections I and K with the tangent. The chords OI and OK are thence given, and being contained in a semicircle, they are mutually at right angles, and therefore mark the position of the two axes. From C draw CL and CM parallel to OK and OI; the semitransverse axis OA, being (Prop. XX. cor. 2.) a mean proportional between OL and OI, is given; and so is the semiconjugate axis OB, being a mean proportional between OM and OK.

COMPOSITION.

Make OC to OD as OD to the semiparameter of the diameter CO*c*, and place this third proportional from C, either in the opposite (fig. 69.) or in the same (fig. 70.) direction with the centre O; bisect OE by the perpendicular GH, which produce to meet the straight line ICK drawn parallel to OD; from H, with the radius HO, describe a circle cutting ICK in the points I and K; join OI and OK, and let fall upon them the perpendiculars CL and CM; between OL, OI and OM, OK, find (VI. 16. El.) the mean proportionals OA and AB: these are the semiaxes required.

For IC.CK = (III. 32. El.) CO.CE = OD^2, and the oblique line ICK, being parallel to OD, is a tangent to the curve, and consequently intersects the axes in I and K. But OA is one of the semiaxes, because, by construction, OA^2 = OL.OI; and for a like reason, OB is the other semiaxis.

But two conjugate diameters being given, the curve may be traced by a succession of points from another investigation, which will afford also a very commodious mechanical construction.

1. *In the Ellipse.*

ANALYSIS.

From D (fig. 72. and 73.) the vertex of the longer semidiameter, let fall upon the other semidiameter the perpendicular DE, on which take DP, either within (fig. 73.) or without (fig. 72.) it, equal to this shorter semidiameter OC, and join OP with the centre; in the diameter DO*d* assume any point I, through which draw GIH parallel to OC, and meeting the curve in the points G, H, and draw IKL parallel to DE, intersecting OP and OC in K and L, and join GK, which produce to meet OC in M. Because (Prop. XIX.) OD^2 : DI.I*d* or $OD^2—OI^2$: : $OC^2 : IG^2$, and the diverging lines OD and OP are cut proportionally to the transverse parallels DP and IK, therefore $DP^2 : DP^2—IK^2$: : $OC^2 : IG^2$; but, by construction, DP = OC, or $DP^2 = OC^2$, and consequently $DP^2—IK^2=IG^2$, or $IG^2+IK^2=DP^2$. Now IK being parallel to DE, which is perpendicular to CE or to GIH, the triangle GIK is right angled, and therefore (II. 10. El.) $IG^2+IK^2=GK^2 = OC^2$; whence GK, being thus equal to the shorter semidiameter, is given. Again, from the property of parallel and diverging lines, DP : PE : : IK : KL : : GK : KM; but GK is equal to DP, and therefore KM is equal to PE, that is, to the difference or sum of the perpendicular DE and the shorter semidiameter OC. And since OP and OC are both given in position, the point K is given, and consequently M, and likewise G, a point in the curve.

COMPOSITION.

Having let fall the perpendicular DE upon the shorter diameter, make DP equal to the half of it, and draw OP, in which take any point K, and inflect KM equal to the sum (fig. 72.) or the difference (fig. 73.) of DE and OC, and in KM or its extension make KG equal to OC; then will G be a point in the curve. For, parallel to DE, draw IKL, meeting DO*d* in I, and join GI. Because IK : KL : : DP or GK : PE or KM, therefore IG is parallel to OC, and the triangle GIK right angled; but since OD : OI : : DP : IK, it follows that $OD^2 : OD^2—OI^2$ or DI.I*d* : : DP^2 or $OC^2 : DP^2—IK^2$, that is, $GK^2—IK^2$ or IG^2, and alternately $OD^2 : OC^2$: : DI.I*d* : IG^2; wherefore IG being parallel to the diameter OC, is an ordinate to the curve.

Scholium. It hence appears, that if the position of OP be determined by a previous construction, and an inflexible line GKM, having its segments equal to those of the perpendicular DPE, be turned about so that the point K shall move along OP, and M along OC*c*, the intermediate or extreme point G will describe an ellipse. The rectangular *trammel* described in the scholium to Proposition VIII, was therefore dependant on a particular case of this very general construction.

2. *In the Hyperbola.*

ANALYSIS.

Let (fig. 75.) ON and OS be the asymptotes, and, limited by these, draw the tangent NCP, which is consequently parallel to OD; and its segments CN, and CP, being each likewise equal to OD, are given in position and magnitude; wherefore the points N, P, and the asymptotes ON, OP, are given. Through C draw any other straight line QCRS; and the intercepted portion RS, being equal to CQ, is consequently given, and thence the point R in the curve.

COMPOSITION.

Through C draw NCP parallel to OD, and make CN and CP equal to it; join ON and OP, which will be asymptotes to the curve; and having drawn any straight line QCRS, make the segment RS equal to CQ; then will R be a point in the curve, as it is evident from the property of the asymptotes.

BOOK II.

PROP. I. PROB.

A line of the second order being given, to find two conjugate diameters which shall contain a given angle.

If this angle be a right angle, the conjugate diameters are evidently the two axes themselves. In any other case, let it be required in the given curve (fig. 79.) to discover the equal conjugate diameters C*c* and D*d*.

ANALYSIS.

Assume in that curve any point E, draw GHE and GI*e* parallel to those diameters, and join OE and O*e*. It is obvious that GHOI must be a rhomboid, having its opposite sides equal; and since the chords EG and G*e* are parallel to their conjugate diameters D*d* and C*c*, they are ordinately applied, or bisected by those diameters. Whence EH is equal to HG, which is equal to OI, and again HO is equal to GI or to I*e*, but (I. 22. El.) the angle EHO is equal to EG*e*, and consequently to the angle OI*e*; wherefore (I. 3. El.) the triangles HOE and I*e*O are equal, and the angle HEO being equal to IO*e*, the lines EO and O*e* form one straight line, which is consequently a diameter. The points E and *e* are hence both given, and the angle EG*e* being equal to the opposite angle DOC of the rhomboid, the mutual inclination of the conjugate diameters is given; wherefore (III. 16. El.) the locus of the vertex is a given circle, and consequently the intersection G of this with the curve is given; and GE, G*e* being thus given, the parallel diameters DO*d* and CO*c* are likewise given.

COMPOSITION.

Draw any diameter EO*e*, on which describe (III. 28. El.) a segment of a circle containing an angle equal to the given inclination of the conjugate diameters, and meeting the curve in a point G, join GE, G*e*, and parallel to these draw DO*d*, CO*c*; these are the conjugate diameters required. For the angle DOC which they contain, is evidently equal to EG*e*; and since HO bisects E*e*, it likewise (VI. 1. El.) bisects GE, which is therefore ordinately applied to CO*c*, and consequently DO*d*, being parallel to it, is a conjugate diameter.

Scholium. It is obvious that the arc of the circle will cut the curve in two points, except in the case of contact, or when the diameters are equally inclined to the transverse axis. Conjugate diameters forming a given angle may, therefore, have in general two distinct positions.

In the ellipse, the angle contained by them cannot be less than the acute angle formed by the chords joining the extremity of the transverse axis with the extremities of the conjugate; but in the hyperbola, no limitation exists.

In the parabola, the construction of the problem is simpler. The inclination of an ordinate to the diameter or to the axis, will in this case represent the angle which would have been formed by the conjugate diameters. If from the vertex of a parabola, therefore, a chord be drawn at the given angle with the axis, the diameter bisecting it is the one required.

PROP. II. PROB.

An ellipse being given, to find two equal conjugate diameters.

Let COc and DOd (fig. 78.) be the required equal diameters of an ellipse.

ANALYSIS.

From A, one extremity of the transverse axis, draw AB parallel to DOd, meeting the curve in the point B, and join BOb. Because equal diameters must evidently be inclined equally to either axis, the angle COA or EOA is equal to DOA, that is to (I. 22. El.) OAE, and hence (I. 10. El.) EA is equal to EO; but AB, being parallel to the diameter DOd, is consequently bisected by its conjugate COc in E, and the straight lines EA, EO, and EB are all equal. Wherefore the angle AOB, being thus contained in a semicircle, is a right angle, and BOb is hence the conjugate axis. Join Ab; and since the angle OAb is evidently equal to OAB, it is equal to AOC, and COc is parallel to Ab. The equal conjugate diameters COc and DOd, being therefore parallel to the chords AB and Ab, are given in position.

COMPOSITION.

Join the extremities of the axes AOa and BOb, by the straight lines AB and Ab, parallel to which draw the diameters COc and DOd; these are equal and conjugate. For the angle EOA being equal to OAb or to EAO, consequently EA is equal to EO. Again, the angle EOB being equal to the vertical angle bOc, is equal to BOd or to

EBO, and therefore EO is equal to EB. Whence EA is equal to EB, and BEA is ordinately applied to COc, which is therefore conjugate to the parallel diameter DOd. But these diameters are likewise equal, for they are equally inclined to both the axes.

PROP. III. THEOR.

Two tangents being applied to a line of the second order, if from any point in one of them a straight line be drawn parallel to the other, the portion of it intercepted by the chord joining the points of contact will be a mean proportional to its segments made by the curve.

Let (fig. 59.) AT and BT be two tangents to the curve, and CDEd drawn parallel to AT, cutting in E the chord AB which joins the points of contact; the intercepted segment CE is a mean proportional between CD and Cd. For, from the property of parallel and diverging lines, AT : BT : : CE : BC, or $AT^2 : BT^2 :: CE^2 : BC^2$; but (I. 17. cor. 2.) $AT^2 : BT^2 :: DCd : BC^2$, and consequently $CE^2 : BC^2 :: DCd : BC^2$; whence $DCd = CE^2$, and (V. 6. El.) CD : CE :: CE : Cd.

Cor. 1. Hence, if a third tangent IHG be drawn parallel to AT, the segment HI will be equal to HG. For the points D and *d* coalescing in I, it follows, that $HI^2 = HG^2$, or HI = HG.

Cor. 2. Hence in a parabola (fig. 60.), if the tangents AT, BT, and the chord AB meet a diameter LQ in the points M, C, and N; the segment ON will be a mean proportional between OC and OM. For from C draw

CD parallel to the tangent AT, and meeting the production of AB in R. From the property of parallel and diverging lines, CR : AM : : CN : NM, or $CR^2 : AM^2$: : $CN^2 : NM^2$; but, since it has been shown that $CR^2 = DCd$, therefore $DCd : AM^2 :: CN^2 : NM^2$. Again (Prop. XVII, cor. 2.) $DCd : AM^2 :: OC : OM$, and consequently $OC : OM :: CN^2 : MN^2$; whence (V. 26. El.) $OC : ON :: ON : OM$.

Cor. 3. Hence, in the parabola, (fig. 60.) if from any two points A, B, straight lines be inflected to another point K in the curve, the intercepted portions OP, OQ of the diameter will have a given ratio. For draw the tangent KL; and, by the last corollary, BC and KL being tangents, $OP^2 = OC.OL$; and since AM and KL are tangents, $OQ^2 = OM.OL$. Consequently $OP^2 : OQ^2 :: OC : OM$; but the tangents AM and BC, and thence OC and OM, are given, and therefore the ratio of OP^2 to OQ^2, and that of OP to OQ, are likewise given.

PROP. IV. PROB.

To find the *locus* of the concourse of two tangents applied at the extremities of a chord or secant drawn through a given point within or without a given line of the second order.

Let A (fig. 94, 95, 96 and 97.) be a given point through which any chord or secant ABC is drawn; it is required to find the *locus* of the point D, where the tangents at B and C meet.

1. *In the case of the Parabola.*

ANALYSIS.

Through A, (fig. 94. and 95.) draw AE parallel to the axis, meeting the curve in E, at which point apply the tangent EL, and parallel to this draw DG; bisect the chord BC in I, join DI, and draw EH parallel to BI. Because DI, drawn from the concourse of the tangents BD and CD, bisects BC, it is a diameter, and consequently EH is an ordinate applied to it. Wherefore, (I. 20. Schol.) DI, being a subtangent, is bisected at the vertex K, or the absciss IK is equal to KD, and HK to KL, and hence the remainder IH is equal to the exterior segment LD; but the figures GDLE and EHIA being rhomboids, IH and LD are equal to AE and EG; and since AE is given, EG and the point G must also be given. Whence GD, the *locus* of the point D, being parallel to the given tangent EF, is given in position.

COMPOSITION.

Parallel to the axis, draw AE, and produce it if necessary till EG be equal to AE; through C draw GD, parallel to a tangent EL applied at E; GD is the *locus* required. For join DB, DC and draw DI bisecting BC and the parallel EH. KI is evidently a diameter to which EH is an ordinate applied; whence HK is equal to KL; but by construction, AE is equal to EG, and thence IH is equal to LD; consequently IK is equal to KD, and DB and DC are both tangents.

2. *In the case of the Ellipse or Hyperbola.*

ANALYSIS.

Through the given point A, draw the semidiameter OAE, (fig. 96. and 97.) at E apply a tangent EL meeting

OD in L, and, parallel to this tangent, draw DG to meet OE or its extension in G. Because BD and CD are both tangents, OD (I. 17. cor. 3.) must bisect in I the chord BC, which joins the point of contact, and hence EH being drawn parallel to BC, will be an ordinate applied to the semidiameter OK. Wherefore (I. 20. cor. 1.) OI : OK : : OK : OD, and OH : OK : : OK : OL; consequently (V. 6. El.) $OK^2 = OI.OD = OH.OL$, and OI : OH : : OL : OD. Now, from the property of parallel and diverging lines, OI : OH : : OA : OE, and OL : OD : : OE : OG; whence, by identity of ratios, OA : OE : : OE : OG. But OA and OE being given, OG and the point G are (VI. 3. El.) given; and since the tangent at E is given, the parallel GD is likewise given. Wherefore the *locus* of the point D of mutual concourse is a straight line given in position.

COMPOSITION.

From the centre O, draw through the given point A the semidiameter OE, make OG a third proportional to OA and OE, at E apply a tangent EL, and draw GD parallel to it; this straight line is the *locus* required.

For draw OID bisecting BC in I, and EH parallel to BC, and join DB and DC. By construction, OA : OE : : OE : OG; but from the property of parallel and diverging lines, OA : OE : : OI : OH, and OE : OG : : OL : OD. Consequently OI : OH : : OL : OD, and $OI.OD = OH.OL$. Now EH being parallel to the bisected chord BC, is evidently an ordinate applied to the semidiameter OK, and therefore OH : OK : : OK : OF, and $OK^2 = OH.OF$: whence $OK^2 = OI.OD$, and OI : OK : : OK : OD. Wherefore DB and DC are both of them tangents at the points B and C.

PROP. V. THEOR.

If, from any point in the curve, parallels be drawn to two adjacent sides of a given inscribed quadrilateral figure, the rectangles under the segments intercepted by those adjacent and by the other two opposite sides, will have a given ratio.

Let ABCD (fig. 98. and 99.), inscribed in a line of the second order, be a quadrilateral figure, which remains constant, and from any point E in the curve, the straight lines EHK and FEG be drawn parallel to the adjacent sides AB and BC; the rectangle under EI and EH, the segments intercepted by the opposite sides BC and AD, will have a given ratio to the rectangle under EF and EG, the segments intercepted by the other opposite sides AB and DC.

CASE I.—*When the line joining the points A and B is parallel to that which joins C and D.*

For (fig. 98.) extending EI to the curve, the rectangle EH, HK (I. 17. cor. 2.) is to the rectangle BH, HC, as the square of a diameter parallel to AB is to the square of a diameter parallel to BC. But, since AB and CD are by hypothesis parallel, they may be ordinately applied to the same diameter NOP, which must bisect likewise the parallel chord EK; and, diverging to the same point with AD and BC, it must evidently bisect the tranverse parallel HI. Whence the exterior segment EH is equal to IK, and EI to HK; the rectangle EH, HK is thus the same as EH, EI, and the rectangle BH, HC is the same as EF, EG. Wherefore the rectangle EH, EI is to the rectangle EF, EG, as the square of a diameter parallel to AB is to the square of a diameter parallel to BC, that is, in a given ratio.

CASE II.—*When the straight line AB is inclined to CD.*

Produce (fig. 99.) BA and EI to meet the extension of CD in L and M, and join GI and CK.

From the property of parallel and diverging lines, CL : CM : : BL : HM, and LD : MD : : LA : MI; by compounding these analogies, therefore, CL.LD : CM.MD : : BL.LA : HM.MI. But the oblique secants BAL and EKM being mutually parallel, and the secants CL and CM having the same position, it follows, (I. 17. cor. 1.) that CL.LD : CM.MD : : BL.LA : EM.MK; whence, from the identity of ratios, the rectangle HM.MI is equivalent to EM, MK, and therefore (V. 6. El.) HM is to EM, or CM is to GM, as MK to MI, and consequently GI is parallel to CK. Again, the triangles KHC and IEG being hence evidently similar, HK is to HC as IE to EG, and the rectangle HE, HK is to the rectangle BH, HC, as the rectangle HE, EI is to the rectangle BH, EG; but, from the property of the curve, the rectangle HE, HK is to the rectangle BH, HC, as the square of the diameter parallel to AB is to the square of the diameter parallel to CB; whence the rectangle HE, EI is to the rectangle BH, EG or FE, EG, in the same given ratio.

Cor. 1. If the point D (fig. 100.) coincide with A, the chord AD will evidently be represented by the tangent at A. Wherefore EHI and FEG being drawn parallel to AB and BC, and meeting the tangent in I and the produced chords AB and AC in F and G, the rectangle HEI is to the rectangle FEG as the square of the diameter parallel to AB, is to the square of the diameter parallel to BC.

Cor. 2. In like manner, if the point D (fig. 101.) merge in C, the chord CD will pass into the tangent applied at C. Whence EHI and FEG being drawn parallel to AB and BC, and terminating at I and G in AC and the tangent, the rectangle HEI is to FEG in the ratio of the squares of the diameters parallel to AB and BC.

Cor. 3. If the two points D and C (fig. 102.) both merge in A and B, the chords AD and BC will become represented by tangents at A and B; and HIE being drawn parallel to AB, the parallel to CB will be parallel to the tangent at B, the points F and G coalescing in the single point F. Whence the rectangle HEI is to the square of EF as the square of the diameter parallel to AB is to the square of the diameter parallel to the tangent at B.

Cor. 4. If the points C and D (fig. 103.) were conceived to change positions, the quadrilateral figure ABCD would take an inverted form; and consequently EGF and HEI, being drawn parallel to CB and AB, and meeting CD, AB and CD, AD, the rectangle FEG must have a given ratio to HEI. In like manner, if the points B and C (fig. 104.) were interchanged, the quadrilateral figure would be converted into a crucial shape, bounded by a side BC, and the rectangles FEG and HEI would still have a given ratio.

Scholium. In the circle and rectangular hyperbola, the rectangles HEI and FEG are evidently equivalent. If the points of contact A and B (fig. 102.) be infinitely remote, the rectangle HEI will become constant, or the exterior part HE will be inversely as EI; which is the property of the asymptotes to an hyperbola.

PROP. VI. THEOR.

If, from any point in the curve, straight lines be drawn at given angles to two adjacent sides of a given inscribed quadrilateral figure, the rectangle under the segments intercepted by those adjacent and by the other two opposite sides will have a given ratio.

From a point E in the curve, (fig. 105.) let MEN and EKL be drawn to meet at given angles the opposite sides of the given inscribed quadrilateral figure ABCD; the rectangle under the segments EM, EN intercepted by the sides AB and CD, will bear a given ratio to the rectangle under EK, EL, the segments intercepted by the other sides AD and BC.

For draw EHI and FEG parallel to AB and BC. It is evident that the triangles EIL and EHK, having all their angles given, must be given in species. Wherefore the ratio of EH to EK, and that of EI to EL, are both given; and hence the compound ratio of the rectangles EH, EI to the rectangle EK, EL is given. For the same reason, since the triangles FEM and GEN are given in species, the compound ratio of the rectangle FE, EG to ME, EN is given. But, by the last proposition, the rectangle HE, EI has to the rectangle FE, EG a given ratio; whence the rectangle ME,EN, has likewise a given ratio to the rectangle EK, EL.

Cor. 1. Let the sides AB and CD (fig. 106.) of the quadrilateral figure be produced to meet in M, from which a chord MEE′ is drawn through the curve; the points M and N now coalescing, the lines EK and EL and E′K, E′L passing through E and E′ will evidently form the same angles with the sides BC and AD. Wherefore $EM^2 : E'M^2 :: LEK : LE'K$.

Cor. 2. Suppose (fig. 107.) the point B to merge in A, and the secant ABM will pass into a tangent. Whence $EM^2 : E'M^2 :: LEK : LE'K$. And if D likewise coincide with C, (fig. 108.) $EM^2 : E'M^2 :: EL^2 : EL'^2$, and consequently $EM : E'M :: EL : E'L$.

Cor. 3. Let the points C and D (fig. 109.) interchange their positions, and let ML be drawn through L the intersection of AD with BC; then $EM^2 : E'M^2 :: EL^2 : E'L^2$, and consequently $EM : E'M :: EL : E'L$, or the chord EE′ is cut internally in L, and externally in M in the same ratio.

PROP. VII. THEOR.

In a curve of the second order, four points being given, if from the fourth point parallels be drawn to lines joining the first and second, and the second and third, and to meet other lines extending from the first and third to any fifth point —they will have a given ratio.

Let (fig. 110.) A, B, C and D be four points in the curve; join AB and BC, and parallel to them draw DE and DF meeting the lines AP and CP converging from the points A and C to any assumed point P in the curve; then will DE have a given ratio to DF.

Join AD and CD, and through P, draw PHG parallel to BC, and KPI parallel to AB, cutting AB and CD, and meeting the extension of BC and AD; and likewise draw

DFLM parallel to BC. From the property of parallel and diverging lines, KP : KL : : CP : CF : : PH : DF, and IP : DE : : AP : AE : : LM : DM; therefore, by compounding ratios, KP.IP : KL.DE : : DM.DF : LM.PH; but, by the last proposition, the rectangle KP, IP has a given ratio to PH, LM or PG, PH; and, consequently, the ratio of the rectangle KL, DE to DF, DM is given, and since KL and DM are evidently both given, the ratio of DE to DF must also be given.

Cor. 1. Let DE (fig. 111.) be produced to meet the extension of the diagonal AC in Q, and parallel to EF draw QR to meet the extension of DF in R. Since the ratio of DE to DF is given, and their contained angle EDF, or the supplement of ABC, is given, the triangle EDF and consequently QDR, is given in species, and therefore the line CR is given in position. Hence, to assign a point in the curve: Draw any line MN parallel to CR and meeting DQ and DR, and from the points A and C draw the transverse lines AMO and CON, which will intersect in O a point of the curve.

Cor. 2. When NM coincides with CR, the point O falls upon C, and consequently CR is a tangent. In like manner, if FD be produced to meet the extension of CA in S, and ST drawn parallel to EF meeting CD in T; then AT, the final position of AMO, when the point O merges in A, is the tangent at this point.

Cor. 3. If the point B should coincide with A, (fig. 112.) the line DE will evidently become parallel to a tangent applied at A, while DF is parallel to the chord AC. Hence a point O in the curve is now readily found; for having drawn DQ parallel to the tangent at A and meeting AC in Q, and having joined this with R, the point of concourse of the tangent at C with DR a parallel to AC, draw parallel to QR any line MN terminated by the diverging lines EDM and RDN; AM and CN being joined, their intersection will assign a point O in the curve.

Scholium.—The application of this proposition discloses the distinctive features of the several species of the curve. It is evident from fig. 111. that the radiating lines AM and CN must always cross each other, except when M falls on T or Q, and those lines become tangents at A and C; the ellipse hence consists of a single branch returning into itself. The same inference may be gathered from the simpler construction of fig. 112.—In the case of the parabola, fig. 113, the radiating lines AE and CF do not always mutually intersect; and when AE becomes parallel to CF, the point P is thrown to an infinite distance. But this occurs only in a solitary position of the point E, for the oblique inserted line EF has a given ratio to DE. Consequently the parabola consists in a single branch, the sides of which tend constantly to a parallelism.—In the hyperbola fig. 114, the radiating lines AE and CF will, with a certain angle, change from convergence to divergence; but, at the limits on either hand, they will shoot into a parallel direction. Hence the opposite branch of the hyperbola, and hence also the condition of the extreme tangents or asymptotes.

When (fig. 115.) CR is a tangent, and the quadrilateral figure ABCD has its opposite angles equal to two right angles, the curve passes into a circle. But this property may be derived from elementary geometry. For DE and DF being parallel to AD and BC, the angle EDF is equal to the supplement of ABC or to APC (III. 17. El.), and consequently EDFP is contained in a circle, and the angle DEF or DQR is equal to DPF or DAC, that is, (III. 21.) DCR; whence DQCR is also contained in a circle, and the angle DCA or DCQ=DRQ=DFE=DPE or DPA, and the point P lies in the circumference of a circle circumscribing ABCD.

The same conclusion is drawn rather more easily from fig. 116, which corresponds to the second corollary. For the angle DNO=CNR=ACO, and the angle DMO=TAO=ACO; whence the angles DMO and DNO are equal, and DMNO is contained in a circle. But the trapezium DQCR would likewise be contained in a circle, and therefore the angle DOM or DOA=DNM=DRQ=DCQ or DCA; whence the point O lies in the circumference of the circle which passes through D and touches at A and C.

Suppose the three points A, B and C (fig. 117.) to stand in a straight line, the locus of the fifth point P will become likewise a straight line. For DEF, drawn through the fourth point D, is now parallel to ABC; join PD, and produce it to meet CBA in R. It is evident, that RA is to RC as DE to DF, and therefore in a given ratio; and the points A and C being given, AR and the point R are given, and hence DR the locus of P is a straight line given in position.

PROP. VIII. THEOR.

If about two given points as *poles*, two given angles revolve, carrying the mutual intersection of two of their sides along a *directrix* or straight line given in position, the intersection of the other two sides will trace a line of the Second Order.

Let the angles DAG and DCG (fig. 118. and 119.) be supposed to turn about the *poles* A and C; while the intersection of their sides AG and CG moves along a *directrix* GH, the intersection of the sides AD and CD will describe a line of the second order.

CASE I.—*When the given angles turn in the same direction.*

While the angle DAG (fig. 118.) moves into the position PAH, let the angle DCG, turning the same way, advance into the position PCH; the intersection P of their sides AD and CD, or AP and CP will be a point in the curve.

For, from the pole A, draw AB, making the angle DAB equal to the exterior angle AGX; and, from the pole C, draw CB making the angle DCB equal to CGH. Parallel to AB and CB, draw DE and DF, to meet the cross lines AP and CP in the points E and F; and lastly, in DF or its extension, having made DM equal to DE, draw MN parallel to CF and meeting DC produced in N.

Since, by hypothesis, the angle DAG is equal to PAH, add the intermediate angle DAH, and the angle GAH is equal to DAP or DAE; but the angle DAB being made equal to AGX, the supplementary interior angle ADE (I. 30. El.) between the two parallels AB and DE, is equal to the adjacent angle AGH. Wherefore the triangles GAH and DAE are similar.

Again, because the angle DCG is likewise, by hypothesis, equal to PCH, add the common angle GCP, and the angle DCP or DCF is equal to GCH; but, BC and DF being parallel, the angle CDF is equal to the alternate angle DCB, which was made equal to CGH. The triangles DFC and GHC are therefore also similar.

Hence AG is to GH as AD to DE, and CG is to GH as CD to DF, or (VI. 1. El.) as ND to DM or DE;

and therefore (V. 16. El.) AG is to CG as AD to ND. But AG and CG being both given, their ratio is given, and consequently the ratio of AD to ND is given; now the ratio of AD to CD is given, since these lines themselves are given; whence the ratio of ND to CD, that is, the ratio of DM or DE to DF is given. Wherefore, by the last proposition, the intersection P of the given angles DAG and ACG is a point of the curve, which likewise passes through the four points A, B, C and D.

CASE II.—*When the given angles turn in opposite directions.*

While the angle DAG (fig. 119.) moves into the position PAH, let the angle DCG turning the contrary way, come into the position PCH; their intersection P will be a point in the curve.

For, from the pole A, draw AB, making an angle DAB equal to AGH; and, from the pole C, draw CB, making an angle DCB equal to CGX, the angle adjacent to CGH. From D draw DE and DF parallel to AB and CB, meeting AP and CP in the points E and F; and lastly, having made DM equal to DE, draw MN parallel to AB.

Because the angle DAG is, by hypothesis, equal to PAH, take away the common angle DAH and the angle GAH is equal to DAP or DAE; but the angle AGH, being equal by construction to DAB, is equal to the alternate angle ADE. Whence the triangle AGH is similar to ADE; and, in like manner, the triangle CGH is proved to be similar to CDF. Wherefore AG is to GH as AD to DE, and CG is to GH as CD to DF, or as ND to DM or DE; and consequently AG is to CG as AD to ND. But the ratio of AG to CG is given, and hence that of AD to ND; and since the ratio of AD to CD is given, the ratio of ND to CD, or that of DE to DF is given. Whence P, the intersection of the given angles DAG and DCG, is a point in the curve which passes through the four points A, B, C and D.

Cor. 1. When the side AD of the angle comes into the position of the tangent AT, the corresponding side CD of the angle ACD must evidently coincide with CA. Hence, to apply a tangent at A, make the angle ACI equal to DCG, join AI, and draw AT, making the angle IAT equal to DAG, and AT is the tangent required. In the same manner, a tangent may be applied at C, the side AD then passing through that point.

Cor. 2. If one of the revolving angles DAG passes into a straight line (fig. 120.), the formation of the curve will become much simpler.

Cor. 3. If the directrix GH were supposed to be removed to an indefinite distance, it is evident that AG and CG and AH and CH would respectively maintain a parallelism; consequently the angle DAP would be equal to DCP, therefore the curve ADPC would become an arc of a circle, which includes the other portion ABC.

PROP. IX. PROB.

Through five given points, to describe a line of the Second Order, by the revolution of angles.

Let A, B, C, D and P (fig. 121.) be five given points, through which it is required to trace the curve, by means of a combined angular movement.

ANALYSIS.

Let the points A and C be two poles about which the angles DAG and DCG revolve, and while the intersection of the sides AG and CG glides along the directrix GH, the other sides AD and CD carry their intersection in the curve. When the angle DAG moves into PAH, the angle DCG will pass into PCH. From the point D and parallel to AB and BC, draw AE and DF meeting AP and CP in E and F.

By the last proposition, the triangles GAH and GCH being similar to DAE and DCF, are therefore given in species. Whence the ratio of AG to GH, and that of CG to GH are given, and consequently the mutual ratio of AG to CG is given; but the contained angle AGC, or the difference between AGH and CGH being given, the triangle CAG is given in species; and it is also given in magnitude, because its base AC is given. In like manner, the triangle CAH is proved to be given in species and magnitude. Wherefore the points G and H are given; and hence the describing angles DAG and DCG, and the directrix GH are all given.

COMPOSITION.

Assume any straight line *gh*, and upon it describe the triangles *gah* and *gch* similar and similarly placed to DAE and DCF, join *ac* and upon AC construct (VI. 23.) the quadrilateral figure GACH similar in its form and position to *gach*, and having drawn AG and CG, let the angles DAG and DCG turn about the poles AC; and while the intersection G of their sides AG and CG is made to slide along the directrix GH, the intersection of the other sides AD and CD will trace out the curve.

For let these angles come into the positions PAH and PCH, corresponding to any fifth point P. This is a point in the curve, because the triangles GAH and GCH being similar to *gah* and *gch*, are therefore similar to DAE and DCF.

PROP. X. THEOR.

If three straight lines revolve about three given *poles*, and carry two of their mutual intersections along two given *directrices*, their third intersection will describe a curve of the Second Order.

Let APG and LFG, (fig. 122.) radiating from the given poles A and L, carry their intersection G along a given directrix DG, while the intersection F of LFG, with a line CPF radiating from a third given pole A, is carried along another given directrix DF; the intersection of APG with CPF will trace the successive points P of the curve.

It is obvious that the two poles A and C must always be points in the curve; but the third pole L may occupy a position either within or without it.

This construction has evidently two extreme limits when the intersections F and G of the directrices are thrown to an indefinite distance. As the intersection F becomes more remote, the radiating lines LF and CF will be less inclined to DF, and at the moment it vanishes in the distance they will assume the parallel positions LM and CB; the radiating line AM being extended to meet CB, will hence assign the point B. Again, when the intersection G shoots into indefinite remoteness, the radiating lines LF and AG will come into the positions LN and AK parallel to the directrix DG; wherefore the radiating line CN, being produced to meet AK, will mark the point K.

Draw DE parallel to MAB and meeting AG in E, join AD, CD and EF.

Since DE and DF are parallel to MA and ML, it is evident that GD is to GM as DE to MA, and as DF to ML, and consequently DE is to DF as MA to ML. But MA and ML being given lines, their ratio is given, and hence the ratio of DE to DF is likewise given. Now, ABCD being a given quadrilateral figure, and the parallels DE and DF to the sides AB and AC having a given ratio, the intersection of AE with CF is (Prop. VII.) a point of the curve which circumscribes that figure.

Cor. 1. When the radiating line AL passes through the pole L, its intersection H, with the directrix DF, will be a point in the curve, since CH is the line radiating from C. For the same reason, the radiating line CIL will cross the directrix DG at a point I in the curve.

Cor. 2. When the variable point P merges in C, the radiating line from that pole must become tangent. Wherefore, produce AC to meet DG in Q, join LQ intersecting DF in R; the line CR being joined will touch the curve at C. For, in like manner, if CA be produced to meet the directrix in S, and LS joined cutting DG in T, the radiating line AT will be a tangent at the pole A.

PROP. XI. PROB.

Through five given points, to describe a curve of the Second Order by the intersections of three revolving lines.

Let A, B, K, C and D (fig. 123.) be five points, through which it is required to trace the curve, by the operation of lines turning about three poles.

Assume the first and fourth points A and C as two of the poles, and the fifth point D as the vertex of the two directrices; these directrices and the remaining pole will be thus discovered. Join AK and CB, and, parallel to them, draw DG and DF, which will be the directrices. Again, having joined BA and KC, produce them to meet the directrices in the points M and N, and draw ML and NL parallel to DF and DG; their concourse L will mark the third pole. If now, from that point, any line LFG be drawn to cut the directrices in F and G, and these points be connected with the poles C and A; the intersection of the lines APG and CPF will assign a point P of the curve.

For when the point G shoots into the indefinite distance, the radiating lines LG and AG become parallel to the directrix DG, and therefore assume the positions LN and AK, while the line radiating from C takes the position NCK. Again, when the point F vanishes into extreme remoteness, the radiating lines LF and CF acquire the positions LM and CB parallel to DF, while the line AG, which turns about the pole OA, comes into the position MAB. Parallel to this draw DE, and the lines DE and DF are hence parallel to the sides AB and BC of a given quadrilateral figure ABCD, and have evidently the given ratio of MA to ML; wherefore, by the seventh proposition of this Book, the point P must occur in the curve which circumscribes that figure, and likewise passes thro' the point K.

But the solution may be simplified. Let A, H, D, I and C (fig. 124.) be five points, through which it is required to trace the curve, by the revolution of radiating lines. Assume the first and fifth points A and C for two of the poles; and to find the third pole, join the first and second points A and H, and the fifth and fourth points C and I, and produce the lines to meet in L. Lastly, join the third point D with the second and fourth points H and I, to form the directrices DF and DG. It is obvious, that when the line radiating from the pole L occupies the position LN parallel to the directrix DG; the line radiating from A becomes likewise parallel to DG, and is intersected in K by the line NCK radiating from C. In like manner, when the radiant from L assumes a position LM parallel to the directrix DF, the radiant CB being also parallel to DF, is cut in B by MAB, which turns about the pole A. The radiant from L assigns likewise the points H and I. Hence these points belong to the same system with the points B and K, and the poles and directrices are thus the same as those derived from the five points A, B, K, C and D.

Cor. 1. If the point H should coincide with A, the line AH will evidently be a tangent; and, for the same reason, if I should coincide with C, the line CIL will be a tangent at this point.

Cor. 2. If the varying point P merge in C, the radiant from this pole will become a tangent, while the radiant from the pole A will pass through C. Wherefore, produce AC to meet the directrix DG in Q, and draw LRQ cutting the other directrix in R, join CR, and it will touch at C. In like manner produce CA to meet the directrix DF in S, and AT being drawn to the intersection of LS with the directrix DG, will be a tangent at A.

Cor. 3. If the points H and I interchanged their positions, and radiants AI and CH were drawn to cross in L′, this point lying within the curve would have all the properties of the third pole. The radiant CPG′ being drawn to the directrix DG and L′G′, joined and produced to meet the directrix DF in F beyond S, the other radiant F′AP will cut CPG′ in the point P.

PROP. XII. PROB.

By the system of revolving lines, to describe a curve of the Second Order, that shall touch a straight line given by position at a given point, and also pass through three given points.

Let it be required, by the combination of radiants, to trace the curve (fig .125.) through a point of contact A in the line AL, and likewise through the points D, I and C.

Assuming, as in the last proposition, the extreme points A and C as poles, the remaining pole will be easily found. For, on the supposition that there had been originally five given points, the second point H has merged in the point of contact A, and the chord AH has consequently become changed into a tangent. Join CI, and produce it to meet AL in the point L, which is therefore the third pole. The point D being now joined with A and with I, will evidently form the two directrices ADF and DIG.

To find any point in the curve, from L draw a straight line LFG, cutting the directrix AD in F, and the directrix DI in G, and join the lines CPF and APG; their intersection P will mark cut the curve.

Cor. 1. If the chord CI (fig. 126.) be parallel to the tangent at A, the pole L will evidently be thrown to an infinite distance, and the radiants proceeding from it will become all parallel. Let AO be drawn to bisect CI, and it must consequently be a diameter to which that chord is ordinately applied. To find the other extremity P of the diameter, produce it to meet DI in G, draw GF parallel to the tangent, and meeting AD in F, and join CPF.

Cor. 2. If the point D coincide with I, the directrix DI will become a tangent IT, and the directrix AD will migrate into the chord AI. Wherefore, TF′ being drawn parallel to the tangent at A, the cross line CPF′ will terminate the diameter. Now, from the property of parallel and diverging lines, OP : PT : : OC or OI : TF′ : : OA : OT; and hence the interval OT between the chord IO and the tangent IT is divided internally at P, and externally at A, in the same ratio, as was proved in the first Book.

PROP. XIII. PROB.

To describe, by the system of radiants, a curve of the Second Order, that shall form given contacts with two given tangents, and also pass through another given point.

Let AL and CL (fig. 127.) be two tangents given in position, and A and C given points in them, and let it be required, by the revolution of combined radiants, to trace the curve through these points of contact, and likewise through a third given point D.

On the supposition of five original points, the second and fourth H and I may be conceived to have merged into the points of contact A and C. Wherefore, assuming A and C as two poles, the concourse of L of the tangents will mark the third pole. Join the point D with A and C, to form the two directrices.

Any point of the curve is now easily found; for draw LFG to cut the directrices in F and G, and from their points draw lines FCP and GPA to the poles C and A, and their intersection P will define the curve.

Cor. 1. If the point D (fig. 128.) should lie on the other side of the chord AC, the opposite sides AP and CD of the quadrilateral figure ACDP concur in the point G, and the diagonals AD and CP cross it in F in the same radiant LFG. But if the points D and P were assumed as poles, the third pole L assigned by the tangents DL and PL must still occupy the same line. In like manner, in fig. 127. the opposite sides AD and PC, and AP and DC meet in F and G, while the opposite tangents AL and CL, and DM and PM meet in L and M, all these points of concourse ranging along the straight line LG.

Cor. 2. If the poles A and C (fig. 129.) should occupy the extremities of a diameter, the tangents at those points will become parallel to the conjugate diameter, and their intersection L will consequently be thrown to an indefinite distance. Suppose the point D to terminate the conjugate diameter, and draw the directrices AD and CD. Any ordinate NP will be hence found. For extend the parallel radiant FGM, which should have issued from L, and join AF and CG crossing in P, and draw PN parallel to DO. From the property of parallel and diverging lines, $AO : AM :: OD : MG$, and AO or $OC : MC :: OD : MF$, and by composition $AO^2 : AMC :: OD^2 : MG.MF$; but $AM : AN :: MF : NP$, and $MC : NC :: MG : NP$; and by composition $AMC : ANC :: MG.MF : NP^2$; wherefore, by identity of ratios, $AO^2 : ANC :: OD^2 : NP^2$, the general property of the curve.

Scholium. The construction of this proposition indicates very clearly, by its variation, the distinguishing features of the several lines of the Second Order. Let the transverse axis (fig. 130.) bisect AC at right angles, and consequently pass through the concourse of the tangents AL and CL; D, the vertex of the curve and of the directrices lying in the same line, suppose it first to be nearer to O than to L. Since OD : DL : : OA : LF, it follows that OA is less than LF, and consequently the radiants FA and GC, which limit the axis, will meet towards E and make an ellipse. But if, as in fig. 131., DO be equal to DL, then OA will be equal to LF, and the radiants FA and GC will run parallel to the axis, and never meet it, indicating thus a parabola. If the radiant FLG change its position ever so little into F′LG′, the radiants F′A and G′C will meet; for, through G draw RGS parallel to FC, and the triangle GLR is evidently equal to FLF′. But RG : GS : : DF′ : DC; and DC being greater than DF′, therefore GS is greater than RG or FF′. Comparing now the triangles FAF′ and GCS, the angle GCS must be greater than FAF, and hence the line GC is inclined to FA, and would meet it at some distance.

Again, suppose DO (fig. 132.) to be greater than DL; and OA being therefore greater than LF, the radiants AF and CG must meet at E on the other side of D, thus intimating the hyperbola. But it will appear from the inspection of the figure, that the radiants AF′ and CG′ will at first continue to meet in the branch of the curve which has E for its vertex, and will, after passing a certain limit, concur in the opposite branch.

Let CA (fig. 133. and 134.) represent a diameter of the curve, of which O is the centre, and D the extremity of the conjugate. The tangents at C and A being parallel, the pole L must vanish into distance. The directrices CD and AD must, in the ellipse, meet from opposite sides; but, in the hyperbola, they will run parallel to each other, tending to the indefinitely remote extremity of an imaginary conjugate. Let the parallel radiant FGN, which should proceed from L, cut those directrices in F and G, and the cross lines AG and CF will mutually intersect in a point P of the curve.

Now CO : CM : : OK : PM, and CO or AO : MA : : OQ : PM; whence, by composition, AO^2 : CMA : : OK.OQ : PM^2. But NF : NG : : OK : OD : : OD : OQ, and therefore OK.OQ = OD^2, and consequently AO^2 : CMA : : OD^2 : PM^2, the noted property of those curves.

The parabola may be considered as an hyperbola of extreme elongation. Let, therefore, (fig. 135.) the diameter shoot into indefinite distance, and the radiants from A will become parallel. But the directrices must consist of AFD′ in a given position parallel to the diameter, and of CGD′, which makes a given angle with it. If, therefore, the parallel radiants APG and FGN be drawn through any point G in CD′, the oblique radiant CPF from C will cut AG in P, a point of the parabola.

Now, CM : CN : : PM : NF, and CN : CD″ : : NG or PM : D′D″ or NF; wherefore, by composition, CM : CD″ : : PM^2 : $D'D''^2$, the distinguishing property of the curve.

To discover the principal features of the hyperbola, let the poles A and C (fig. 136.) vanish into distance, and the tangents LA and LC will become asymptotes. If D be any point in the curve, draw DX and DY parallel to LY and LZ, and they will form the directrices. Let the radiant LGF cut these in G and F, and the parallels GP and FP will assign the point P. Complete the figure, and the parallelogram GZ is evidently equivalent to GY; whence the inscribed parallelogram PL is equivalent to DL or a given space, a noted property of the asymptotes.

If the lines AD, CD (fig. 137.) inflected from the extremities of the diameter to terminate the perpendicular diameter be at right angles, the ellipse will migrate into a circle. For the opposite tangents at A and C being parallel, the pole L must be thrown to an indefinite distance, and consequently its radiant GFQ will be perpendicular to CA. Join AG and CF, and their intersection P will be a point in the curve. For in the triangle CGA, the lines AD and GQ are perpendicular to the sides CG and CA, it follows that CP is likewise perpendicular to AG; and the angle CPA being thus a right angle, the point P must lie in a semicircle. Again, from similar triangles, CK : PK : : PK : KA, and hence $PK^2 = CKA$.

In the hyperbola, the conjugate axis is thrown to indefinite distance. When the curve is equilateral, the directrices CG′ and AF′, instead of being perpendicular as in fig. 137., run parallel, and with the inclination of half a right angle to the transverse axis. The perpendicular radiant from L therefore cuts those lines in G′ and F′, and the intersection of AG′ and CF′ marks a point P′ of the curve. But CQ and AQ being evidently equal to G′Q and F′Q, the triangles CQF′ and G′QA are equal, and consequently the angle ACP is equal to AP′K′. Wherefore the triangles CK′P′ and P′K′A are similar, and CK′ : P′K′ : : P′K′ : AK′, and hence $CK'A = P'K'^2$.

PROP. XIV. PROB.

To describe, by the system of radiants, a curve of the Second Order, that shall form given contacts with two given tangents, and also touch another straight line given in position.

Let AL and CL (fig. 138.) be two given tangents, with the points of contact A and C, through which it is required to trace, by the revolution of combined radiants, a curve of the Second Order that shall likewise touch the line KI.

Assuming, as in the last Proposition, the points A, C and L as three poles, it is only wanted to discover, in the tangent KI, a point D, through which the two directrices AD and DC pass. Now, since the curve touches at D, any contiguous point P of the curve must evidently lie in the direction of the tangent. But, in that limiting position, the radiants from the poles A, C and L, would still retain their parallelism to AD, CD and LD. Consequently the parallelogram FDGR being described, having its diagonal FKG parallel to LD, the tangent DKR must form the other diagonal. These diagonals are mutually bisected in K, and therefore GK : DL : : FK : DL; whence AK : AL : : KS : SL, and the line AL is harmonically divided in the points K and S. But since AO is intersected by the lines drawn from D to those points of harmonic section, it is likewise (VI. 8. Geom.) cut harmonically in Q and C. For the same reason, L being a radiating point, the line AO is divided harmonically in D and I, or KO : OI : : KD : DI. But the points A, I and O being given, the point D is likewise given; and the proposition is thus reduced to the conditions of the preceding problem.

Cor. 1. It hence appears, that a tangent KI between two other tangents, is divided internally at the point of contact D, and externally at O, where it meets the line joining the two contacts A and C in the same ratio. This conclusion might also be derived from Prop. XX. Book I.

Cor. 2. If the tangent KI run parallel to the chord AC, it must evidently be bisected in the point of contact, and the radiant LD will consequently pass through the centre of the curve. In the case of the hyperbola, when the points A and C vanish in the distance, the tangents AL and CL become asymptotes, and the intercepted tangent KI is always bisected in D.

PROP. XV. THEOR.

If, in a curve of the Second Order, a hexagon be inscribed, and each pair of its opposite sides be produced to meet, the three points of concourse will lie in the same straight line.

Let a hexagon ABCIDH (fig. 139.) be inscribed in the curve, and produce the opposite sides BA and ID, AH and CI, HD and BC, to meet severally in the points N, L and M; these points of concourse will occupy the same straight line.

For, conceive the points A, H, D, I and C to be five original points; if the points A and C be assumed as poles, the chords AH and CI will meet in L, the third pole, and the extended chords HDM and IDN will form the two directrices. Wherefore the radiant BAN, which proceeds through the pole A to a sixth point B, must cut the directrix IDN in N, and the radiant from the pole C to the same point must cut the directrix HDM in M, while the radiant which turns about the pole L must pass through both these intersections N and M.

Cor. 1. If the point B were to change its position into B′, the concourse of HD and CB would mark the point M′, and the concourse of DI and AB′ would mark the point N′, both of them lying in the new radiant LM′N′.

Cor. 2. If the point H coincide with A, the chord AH will pass into a tangent. Whence the sides BA and ID, and the sides AD and BC of the inscribed pentagon, meet in the points M and N, while the tangent at A and the side CI meet in L, these points occupying the same radiant MLN.

Cor. 3. If both the points H and I (fig. 140.) should merge in A and C, the hexagon will pass into the quadrilateral figure ABCD. Wherefore the opposite sides BA, CD and AD, BC, and likewise the opposite tangents AL, CL, and BL′, DL′ concur in the points N, M, L and L′, which lie all in the same straight line.

Cor. 4. Conceive all the points of concourse to be thrown to an indefinite distance, and let the points A and B coincide, while C, H and I change their positions; it will then follow that BC and BH (fig. 141.) must be parallel to the DH and CI, and the tangent at A parallel to the chord DI. Hence a simple method of applying a tangent at any point of the curve; for having inflected the chords AC and AH, draw HD and CI parallel to them, join DI, and parallel to it, draw through A a line which must touch at that point.

PROP. XVI. PROB.

To find by a primary construction a line of the Second Order that shall pass through five given points.

Let A, B, C, D and E (fig. 142.) bet he five points through which it is required to describe a curve of the Second Order, without employing the revolution of angles or of lines.

ANALYSIS.

Join A, B, and C, D by two straight lines which meet in H, and A, D, and B, C by two other straight lines meeting in G; and from the fifth point E draw EML and EIK parallel to ABH and CBG, and cutting the curve in M and K, and meeting HCD and DAG in L and K. From the property of the curve, DH.HC : AH.HB :: DL.LC : EL.LM; but the three first terms of this analogy being given, the rectangle EL.LM is given; and since EL is given, LM is also given, and the point M in the curve. Bisect the chords AB and EM in V and T, and the straight line YVTZ, which joins these points, is a diameter given in position. Again, DG.GA : CG.GB :: DI.IA : EI.IK, and consequently the rectangle EI.IK and thence IK and the point K in the curve are given. Bisect the parallel chords BC and EK by the straight line SOR, which will be a diameter given in position. But these diameters YZ and SR are likewise given in magnitude; for $AV^2 : ET^2 ::$ YVZ, or $OY^2—OV^2$: YTZ, or $OY^2—OT^2$, and hence the ratio of $OY^2—OV^2$ to $OY^2—OT^2$ is given; and since OV and OT and their squares are given, the square of OY, and OY itself, are (VI. 2. and I. 23. El.) given. In the same manner it is shewn that OS is given. Again, from the property of parallel lines intersecting the curve, DI.IA : EI.IK :: $OZ^2 : OX^2$, and consequently XOW, the conjugate of SOR is given, and the curve is given, by Proposition XXX. of the last Book.

COMPOSITION.

Draw AB, DC to meet in H, and CB, DA to meet in G, and through E draw EL and EK parallel to ABH and CBG. Make DHC : AHB :: DLC : ELM; and again, DGA : CGB :: DIA : EIK, and the points M and K will be found. Bisect the chords AB, EM, and BC, EK by the straight lines YZ and SR, which will define the position of two diameters. Next, make $TE^2—AV^2 : AV^2 ::$

$OV^2 - OT^2 : OY^2 - OV^2$, whence OY will be found; and again, make $IK^2 - QC^2 : QC^2 :: OQ^2 - OI^2 : OS^2 - OQ^2$, and OS will be likewise found. Lastly, make DIA : EIK : : $OZ^2 : OX^2$, and thus XW, the conjugate of SOR, is found, and the curve is determined, by Proposition XXX. Book I.

PROP. XVII. PROB.

To find, by a primary construction, a line of the Second Order that shall pass through three given points, and touch two straight lines given in position.

Let FAR and FHB (fig. 143.) be two given tangents, and C, D and E three given points through which it is required to describe a line of the Second Order, without employing the revolution of angles or lines.

ANALYSIS.

Join the points of contact A and B, draw the chord CD and produce it to meet AB in I; connect likewise the chord DE, extend it both ways to meet the tangents FB and FR in H and R, and, lastly, parallel to these draw GKE and HSM.

Since NB and GL are parallel, (I. 17. cor. 2.) CND : DGC : : NB^2 : LGK or $GT^2 :: NI^2 : GI^2$. But the rectangles CND and CGD being given, the ratio of NI^2 to GI^2, and consequently that of NI to GI is given; whereof GN being given, the point I is likewise given. Again, because HM and FR are parallel, EHD : DRE : : MHS or $HQ^2 : AR^2 :: HP^2 : PR^2$; but the rectangles EHD and DRE being given, the ratio of HP to PR is hence given, and the line HR being given, the point P is also given. Wherefore the points I and P being given, the line IBPA is given in position, and consequently its intersections with the tangents, or points of contact A and B are given. The problem is thus reduced to the description of a curve through five given points, A, B, C, D and E.

COMPOSITION.

Find V and X mean proportionals to the rectangles CND and CGD, and Y and Z mean proportionals to the rectangles EHD and DRE. In the ratio of V to X divide ND in I, and in the ratio of Y to Z divide HR in P; join IP, and produce it to cut the tangents in A and B, which are the points of contact, with the curve. The problem is thus brought under the last Proposition.

PROP. XVIII. PROB.

The focus being given, to describe a curve of the Second Order through three given points.

Let it be required to find a curve which passes through the points A and B and C, (fig. 144.) and has its focus placed in F.

ANALYSIS.

Join AB and BC, and produce these chords to the directrix. From the general property of the curve, AF : AD : : BF : BD, and alternately, AF : BF : : AD : BD. But AF and BF being given, their ratios is given, and consequently that of AD to BD; wherefore the points A and B being given, the point D in the directrix is likewise given. In the same way, it is shown that the point E, where the chord BC meets the directrix, is given. Whence the directrix DE itself being given, the curve is determined.

COMPOSITION.

Draw the chord BA, and extend it till FA : FB :: AD : BD; draw likewise the chord BC, and extend it till FC : FB :: CE : BE; now join DE, which will be the directrix to the curve.

If two of the given points, such as A and B, should lie in the opposite branches of a hyperbola, it is obvious that AB must be cut internally in D in the ratio of FA to FB. If two of the points, such as B and C, be equidistant from the focus, the point E will be thrown to an indefinite distance, and consequently the directrix will pass through D parallel to BC. When all the three points A, B and C are equidistant from the focus, the directrix becomes indefinitely remote, and the curve migrates into a circle, having F for its centre.

PROP. XIX. PROB.

By the system of radiants, to describe, through three given points, a curve of the Second Order, whose centre is given.

Let O (fig. 145. and 146.) be the centre of a curve, which it is required to trace through the points A, C, and D.

Join AO and CO, and produce them to equal distances OI and OH, beyond the centre; the extremities I and H of these diameters must evidently be points in the curve. Five points being hence assigned in it, the problem is reduced to Prop. 11.

But the construction may be greatly simplified; the triangles AOH and IOC being obviously equal, the angle OAH is equal to OIC, and consequently the base AH is parallel to CI. Wherefore the third pole I vanishes in the distance, and all its radiants must be parallel to AH or CI. To find any point in the curve, draw, between the directrices HD and ID, the parallel radiant FG, and from the poles A and C draw the oblique radiants AG and CF to meet in P.

Cor. 1. When the points F and G are thrown to an indefinite distance, it is evident that the radiants AB and CB will become parallel to the directrices ID and HD. Hence the opposite pairs of the sides of the inscribed hexagon ABCIDH are all parallel.

Cor. 2. This proposition will also apply to the parabola, if the direction of the axis be substituted for the position of the centre. For H, D and A (fig. 147.) being three points in the curve, other two points I and C, indefinitely remote, may be conceived to occur in the extension of the axis. Consequently, the chord HD and a parallel ID to the axis will form the directrices; while A and the remote point C, in the line of the axis, are two poles, and the third pole L is represented by a vanishing point in the direction of the chord HA, or its concourse with the extreme chord IC. Any point of the curve is hence found, by drawing between the directrices the radiant FG from L parallel to HA, and extending to the pole A the line FA to meet the parallel radiant CG in P.

From this construction it follows, that HM : HN :: MD : NG or MF, and MA : NA :: MF : NP; whence, by compounding the analogies, HMA : NHA :: MD : NP.

PROP. XX. PROB.

By the system of radiants, to describe a parabola, of which the direction of the axis, the contact of a tangent, and a point in the curve are given.

Let it be required to trace, by a combination of radiants, a parabola that shall touch AL in A, (fig. 147.) pass through the point I, and have its axis parallel to AE.

Having drawn the double ordinate IED parallel to the tangent AL, and made ED equal to IE, the point D will evidently lie in the curve. Wherefore, on the supposition of five original points in the curve, two of them, A and H, may be conceived to coalesce in the point of contact of the tangent, and a third point C to vanish in the direction of the axis. The lines DAK and DAL will hence form the two directrices; and IL being drawn towards C, or parallel to the axis, to meet the extension of AH or the tangent in L, this point, with A and the remote point C, will constitute the three poles.

To find any point in the curve, from A draw the radiant AP, cutting the directrix DEI in G, extend through L the oblique radiant GLF, and from F draw the parallel radiant FP to intersect AP in P.

But this point, where the oblique radiant AP meets the curve, might also be discovered by another construction: For the ordinate DE being equal to EI, it is evident that IL is equal to LK, and MO equal to OF. Consequently, from the property of parallel and converging lines, GN : NP : : GL : LF : : IL : OF or MO, but IL : MO : : AI : AM ; ;

AN : AP, and, therefore, GN : NP : : AN : AP, and by division AG : AN : : AN : AP, or AG : AN : : AI : AM, and consequently NM is parallel to GI. Wherefore, let AP cut NIL in N, and parallel to the tangent AL, draw NM to meet AI in M; then MP parallel to NIL will, by its intersection, assign the point P.

PROP. XXI. PROB.

Given the direction of each asymptote and three points in the curve, to describe an hyperbola by the system of radiants.

Let it be required to trace, by the combined intersections of lines, an hyperbola through the points E, D and H, (fig. 148.) and having its asymptotes parallel to EL and HL.

It is evident, that, of five original points A, E, D, H, C in the curve, if the extremes A and C were to vanish in the distance, the lines LE and LH, parallel to the asymptotes, would tend towards them. Wherefore L is one of the poles, the remote points A and C representing the other two poles, while the cross lines DE and DH are the corresponding directrices. Consequently, to find any point P of the curve, from the pole L extend any oblique line LGF to meet the directrix DE in F and the directrix DH in G, draw FP the radiant to C parallel to LH, and intersect this by the radiant GP, tending towards A, and parallel to LE.

Cor. When the radiant LGF comes into the position LM parallel to DE, it is evident that the point P vanishes towards A. Wherefore the parallel OM must be an asymptote. In like manner, the other asymptote ON is found, by drawing LN parallel to DH, since the point P would then retire towards C.

PROP. XXII. PROB.

An asymptote being given and three points in the curve, to describe an hyperbola, by the system of radiants.

Let it be required to trace, by the combined intersections of lines, an hyperbola, one of whose asymptotes is AL, (fig. 149.) through the points D, E and C.

Recurring to the original five points A, H, D, E and C, it is obvious that the adjacent points A and H must merge into one, and vanish in the distance along the asymptote. Wherefore CE, being produced to meet that line, will assign the pole L. Again, the oblique line DE will constitute one of the directrices, while DF, tending to the remote point H, and therefore parallel to the asymptote, will form the other directrix.

To find any point in the curve, from the pole L extend the oblique radiant LFG, from F draw FC through the pole C, and intersect it by GP, tending to the remote pole A, and therefore running parallel to the asymptote.

It must likewise follow, that LM drawn parallel to the directrix DE, will determine the position CM of a parallel to the other asymptote.

General Scholium. It hence appears that the application of the system of radiants not only discloses finely the relations of curves of the second order, but affords the most easy and ready methods in practice for solving a variety of problems.

The same theory likewise throws much light on other geometrical constructions, and particularly on the procedure employed in Linear Perspective. Thus, let DF (fig. 150.) be the base of the picture, and CK the horizontal line at the height of the eye, E *the point of sight*, and the perpendicular EA equal to the *principal ray*, or the distance of the eye from the picture; any point G on the ground plan wili be projected in the line CF drawn from *an accidental point* C to join F, where GF, the parallel to AC, meets the base, its place being marked by the intersection of GA. But P may be formed by a different construction; for the oblique line DG being drawn from any point in the base to G, and AK being extended from A parallel to it and DK joined, this line will be the projection of DG and must consequently cross CF in the required point.

The same conclusion is derived from the theory of radiants. Conceive A, C, H and I four of the original points, to occupy a given straight line, while the fifth D lies beyond it; the locus of any sixth point P, determined by the combination of radiants, will be another straight line DKA, given likewise in position. For A and C being two poles, the third pole L will vanish in the extension of ACHI, and DH and DI will form the two directrices. Any radiant FG parallel to AI being therefore drawn, and the oblique radiants FC and GA produced to the poles A and C, their intersection will mark a sixth point P in the same system. For, parallel to DF, draw CK to meet the extension of DP in K, and join AK. From the property of parallel and diverging lines, AC : FG : : CP : PF : : KC : DF, and alternately, AC : KC : : FG : DF; but the angle DFG being equal to FHC, is therefore equal to ACK; consequently (VI. 13. Geom.) the triangle KAC is similar to FGD, and the angle AKC equal to GDF. Wherefore CK being parallel to DH, it follows that AK is parallel to DG. Both the lines AK and CK are hence given in position, and, therefore, the point K and the line DPK are likewise given.

If the point D (fig. 151.) should fall into the same straight line with A and C, and absorb also the points H and I, tangents to these points would form the directrices DF and DG, and the construction would become simpler. In this case DP : PK : : PF : PC : : PG : PA, and, therefore, AK is parallel to DG, and the point K and the line DK given in position.

But if the point I were supposed to shift its place along the line ACD, the directrix DG, and consequently its projection DK, would likewise change their mutual positions. If DG terminate in the circumference of a given circle, DP will trace out its projection, which is always a curve of the Second Order. The resulting curve must indeed be a section which the vertical plane of the picture makes through a cone formed by the oblique rays flowing to the eye from the circle on the ground. The nature of this curve, however, may be deduced from the primary construction. Let A and D (fig. 152.) be given points, and DH and CK parallels given in position.

To avoid unnecessary complication, suppose the circle which is to be represented in perspective to touch the base at O, and having drawn DG to touch the circumference in G, and extended AK parallel to it, join the lines AG and DK, and their intersection P will mark the contact of DP, a tangent to the curve. From any point H, between O and D, draw HQ parallel to DK, join AQ, and parallel to it extend HMN to cut the circle, and join AM and AN, which will cross HQ in the points R and S, where the secant HRS meets the curve. Lastly, produce NH and GD to meet in I, and join ACI.

Since DHQK is, by construction, a parallelogram, the side DH is equal to KQ; consequently the triangles IDH and AKQ, having all their sides mutually parallel, are equal. Wherefore ID is equal to AK, and IH to AQ, and hence the figures IDKA and IHQA are parallelograms. Now, from the property of parallel and diverging lines, GD or OD : PD : : GI : AI, and therefore $OD^2 : PD^2$: : $GI^2 : AI^2$. Again, NH : NI : : SH : AI, and MH : MI : : RH : AI; whence, by composition, NHM : NIM : : SHR : AI^2; but the rectangles NHM and NIM are equivalent to the squares of OH and IG; consequently $OH^2 : SG^2$: : SHR : AI^2, or alternately OH^2 : SHR : : $IG^2 : AI^2$. Again, IG : AI : : GD or OD : PD, and $IG^2 : AI^2$: : $OD^2 : PD^2$; wherefore, by identity of ratios, OH^2 : SHR : : $OD^2 : PD^2$. But OD and PD being tangents, and the parallel HRS a secant, this property designates a curve of the Second Order.

PROP. XXIII. THEOR.

If a circle meet a line of the Second Order, the straight lines joining each pair of the points of intersection are equally inclined to the axis.

Let (fig. 84. and 82.) a circle meet the curve in the four points A, B, C and D; the chords AC and BD will make equal angles with the axes, and so will the secants ABE and DCE.

For draw the semidiameters OK, OL parallel to those chords, and the tangents GI and HI parallel to the secants. From the property of the curve, AT.TC : BT.TD :: OK^2 : OL^2; but since the chords AC and BD cut each other within the circle, AT.TC = BT.TD, and consequently $OK^2 = OL^2$, and OK = OL. Wherefore these equal semidiameters, and their parallel chords, must be equally inclined on each side of the axis. Again, AE.EB : DE.EC :: IG : IH^2; but (III. 26. El.) AE.EB = DE.EC and therefore $IG^2 = IH^2$, and IG = IH, which equal tangents must likewise be equally inclined to the axis IMN.

Cor. Hence in the parabola, (fig. 82.) the perpendiculars let fall from the points of intersection of a circle upon the axis, are together equal on either side. For the perpendiculars AP and BQ are double of the perpendicular from the middle point of AB, and which is equal to GV; and for the same reason the perpendiculars DS and CR are double of the perpendicular from the middle point of CD, and which is equal to HV. Wherefore HG being equal to HV, it follows that AP + BQ = DS + CR.

PROP. XXIV. PROB.

To find two mean proportionals between two given straight lines.

Let AB and AC (fig. 80.) be two straight lines, between which it is required to find two mean proportionals.

ANALYSIS.

Suppose the given lines to be perpendicular, and let AH and AE be the mean proportionals. Complete the rectangle AHDE; and because AB : AH : : AH : AE, or AB : DE : : DE : AE, it follows, (V. 6. El.) that $DE^2=$ AB.AE, and therefore the locus of the point is a parabola, having AB for its axis, and AC for its parameter, and which is therefore given. Again, since AH : AE : : AE : AC, or AH : DH : : DH : AC, it is evident that $DH^2=$ AC.AH, and that the locus of D is likewise a given parabola, having AC and AB for its axis and parameter. Wherefore D, the point of the intersection of these curves, is given, and consequently the mean proportionals AH and AE.

COMPOSITION.

On the axis AH, and with the parameter AC, describe a parabola, and on the axis AE, and with the parameter AB, describe another parabola intersecting the former in D, from which let fall the perpendiculars DH and DE; these are the mean proportionals required. For AB.AE$=$ DE^2 or AH^2, and AB : AH : : AH : AE. Likewise, AC.AH $=$ DH^2 or AE^2, and consequently AH : AE : : AE : AC.

Otherwise thus:

Let (fig. 81.) AB and AC at right angles, be the given lines, and AE and AH the mean proportionals required.

ANALYSIS.

Construct the rectangles ACDB and AEGH. Because AB : AE : : AE : AH : : AH : AC, or AB : AE : : AH : AC, therefore (V. 6. El.) the rectangles ACDB and AEGH are equivalent, and consequently the locus of the point G is an hyperbola passing through D, and having AB and AC for its asymptotes. Complete the rectangle AEGH, draw the diagonal AG, and join DG, which produce to I and L. Since AB : AE : : AE : AH, and, from the property of the hyperbola, DI = GL, or AB = HL; consequently HL : HG : : HG : AH, and the right-angled triangles LHG and HGA are similar. Wherefore the angle AGD being equal (I. 30. El.) to the two acute angles GAH and GLH, or HGA, is a right angle, and the locus of the point G is a semicircle described on the given diameter AD; this point being the intersection of a given circumference with a given curve, is consequently given, and thence the rectangle AG contained by the mean proportionals AE and AH.

COMPOSITION.

Apply a rectangular hyperbola to the asymptotes AB and AC and passing through the point D, draw the diagonal AD, and upon it describe a circle meeting the curve again, in the point G, from which draw GE and GH parallel to the asymptotes; the segments AE and AH are the mean proportionals required; for draw the diagonals AD and AG, join DG, and produce it to meet the asymptotes in the points I and L. Because the angle AGD in the semicircle is a right angle, the triangle AGL is right-angled, and divided by the perpendicular GH; whence HL : HG : : HG : AH; but (Prop. 13.) DI = GL and AB = HL, and therefore AB : AE : : AE : AH. Again, from the property of the hyperbola, AB.AC = AH.AE; whence AB : AE : : AH : AC, and AH and AE are mean proportionals inserted between AB and AC.

Scholium. This proposition embraces the famous *Delian Problem* or the *Duplication of the Cube*, which the ancient Greeks were the more eager to solve geometrically, as they had not, with all their ingenuity, attained sufficient skill to extract the cube root by a process of arithmetic.

PROP. XXV. PROB.

To trisect a given angle.

Let (fig. 83.) *a*OF be an angle, of which it is required to find the third part.

ANALYSIS.

From the vertex O of the angle describe a circle, and the arc FD, which subtends at O the third part of the angle *a*OF, must likewise be the third of the whole intercepted arc FD*a*. Join *a*F, and, parallel to it, draw the chord DG, which (III. 18. El.) will cut off an arc, *a*G equal to FD, and therefore leave the intermediate arc DG, equal to another third part of FD*a*. Join FD, and draw the diameter B*b* perpendicular to *a*F or DG. Because B*b* is at right angles to the chord DG, it (III. 4. El.) bisects this in E, and consequently FD is double of DE. The ratio of the distances FD and DE is thus given, while the point F is given, and CB is given in position; wherefore, the locus of the point D of trisection is a given hyperbola, of which F is the focus, CB the directrix, and the determining ratio that of *two* to *one*. But the circle being given, its intersection D with the curve is likewise given.

COMPOSITION.

Having described the circle, draw the chord *a*F, and let fall the perpendicular OCB, from the focus F, with the directrix CB, and the determining ratio of 2 to 1, describe an hyperbola, cutting the circumference in the point D; the arc FD is the third part of *a*DF, which subtends the given angle.

For draw DG parallel to *a*F, and join FD and *a*G. From the nature of this hyperbola, the focal distance FD, being double of DE, is equal to the chord DG, and consequently the arc FD is equal to DG; but it is likewise equal to the arc *a*G, (III. 20. El.) and therefore is the third part of the whole arc *a*BF.

But, besides the point D, this construction affords likewise two more points of trisection D′ and D″. In the former, since FD′ is double of D′E′, it is equal to D′G′, and consequently the arc FBD′ is equal to D′*b*G′; but the chord D′G′, being parallel to *a*F, the arc FBD′ is also equal to G′B*a*. Wherefore, these three arcs FBD′, D′*b*G′, and G′B*a* are all equal, and they compose evidently the whole circumference, together with the original arc FD*a*; hence the second arc of trisection FBD′ consists of a third part D*a*D′ of the circumference, joined to the first third FD. Again, because FD″, being double of D″E″, is equal to D″G″, the compound arc FBD′*b*D″ is equal to D″FB*a*D′G″; and D″G″ being parallel to *a*F, the arc FBD′*b*D″ is equal to G″*b*D″FB*a*; and, consequently, the three large arcs FBD′*b*D″, D″FB*a*D′G″, and G″*b*D″FB*a* are all equal, and combine to form two circumferences together with the original arc FD*a*. Whence the third arc of trisection FB*a*D′*b*D″, is composed of two-third parts D*a*D′ and D′*b*D″ of the circumference, joined to the first third FD.

The reason of this curious result is, that the arc FD*a*, measuring the angle FO*a*, and formed by the revolution of its radius OF, necessarily involves in it all the successive arcs, which have their origin at F, and their termination at *a*. The describing line OF will again arrive at the position O*a*, after having made one, two, or more circuits; and the intersections of the hyperbola with the circle must exhibit, therefore, the third part of those successively compounded arcs. But though the arcs themselves are thus absolutely innumerable, their trisection gives only three varieties; for each accession of a whole circumference, and which is the result of three added circuits, only returns back on the same points of division.

A similar conclusion may be derived from Prop. 31. of the First Book of Geometrical Analysis; for it was there shown, that if a straight line CFG be drawn from the extremity of the arc AC, such as to intercept, between the circumference and the extended diameter, a segment FG, equal to the radius of the circle, the arc EF will be the third part of AC. But this line CFG is evidently capable of having three several positions, one beyond either end of the diameter, and another crossing it; and it is easily demonstrated, that the points of successive intersection, F, F′, and F″, are mutually separated by intervals, each equal to the third of the whole circumference, or that, if they were connected together, they would form an equilateral triangle.

Another Solution.

Let it be required to find (fig. 153.) CAD, which is a third part of the angle BAC.

ANALYSIS.

Parallel to AD draw BC, to which, meeting AC, erect the perpendicular AF, crossing at F the extension of CB. Make CE equal to BF, bisect CF in O, and join AO.

Because CAF is a right angle, it is contained in a semicircle, of which the middle point O of the diameter must be the centre. Wherefore AO being equal to OC, the angle OAC is equal to OCA, or the alternate angle CAD, the third part of the given angle BAC; consequently the remaining angle BAO, being two-thirds of BAC, must be equal to the compound angle OAD, or its alternate angle AOB; whence AB is equal to BO. But since OF is equal to OC, and BF equal to EC, the part OB must be equal to OE; wherefore BE is equal to the double of AB, which being given in magnitude, the locus of the point E must be a given circumference described from the point B. Again, because the segment CE is equal to BF, the locus of E is a rectangular hyperbola, passing through B, and having AC and AF for its asymptotes. But both the circle and hyperbola being given, their intersection, or the point E, is given, and therefore the line BC is given in position; whence the angle BCA or CAD, the third part of BAC, is given.

COMPOSITION.

Assume any equilateral hyperbola, and from the centre A draw AB, making with the asymptote AC the given angle BAC; from the point B in the curve, with twice the distance AB as a radius, describe a circle cutting the curve in E; join BE, and produce it to meet AC in C; then ACB is a third part of the angle BAC.

For extend BE to the other asymptote AF, bisect it in O, and join AB and AO. Because BE is by construction double of AB, its half BO must be equal to AB, whence the angle BAO is equal to BOA. Again, from the property of the hyperbola, the segment BF is equal to EC, to which add the equal parts OB and OE, and OF is equal to OC; wherefore O is the centre of a semicircle which contains the right angle CAF, and the radius OA is equal to OC. Consequently the angle OAC is equal to OCA, and the exterior angle AOB is double of either of these; the angle BAC, composed of BAO and OAC, is therefore equal to three times BCA, or the angle BAC is a third part of BAC.

It is obvious that the circle described from B may cut the complete hyperbola in four points, the fourth point being always B′, the other extremity of the diameter BAB′. There consequently result three distinct sections of the given angle BAC, corresponding to the points E, E′ and E″. The lines BE, BE′, and BE″ being inflected to these points, will make equal angles about B, and consequently the inclination of BE′ and of BE″ to AC is augmented successively by the third part of four right angles. Thus, the angle BAO is two-third parts of BAC; and it is easy to show, that BAO′ is likewise two-thirds of BAF, which completes the right angle; whence the combined angle OAO′ is two-thirds of a right angle, or the third part of two right angles. But B is the centre of a circle which would pass through the points O, A, and O′, and consequently the angle OBO′ is the double of OAO′, or the third part of four right angles.

Again, the angle BAO″ is equal to BO″A, and double of F″C″A; consequently BAO″, together with the double of AF″O″ or of O″AF is equal to two right angles; that is, three times the angle O″AF and the angle BAF″, or the triple of the angle O′AF′, are equal to two right angles. The combined angle O′AO″ is therefore the third part of two right angles; consequently the angle O′BO″ at the centre of the circle, which would pass through O′, A, and O″, is the third part of four right angles.

This Problem may be somewhat varied, by substituting an arc for the angle to be bisected, as in the next Proposition.

PROP. XXVI. PROB.

To trisect a given arc of a circle.

Let it be required to trisect the arc ABC, (fig. 154.)

ANALYSIS.

Let BC be the third part of ABC, join AB and BC, make the angle CAD equal to BAC, and produce BC to meet AD in D.

It is obvious, from the hypothesis, that the arc AB is double of BC, whence the angle ACB is likewise double of the angle CAB; but (I. 30. El.) ACB is equal to CDA and CAD, and consequently the angles CDA and CAD or CAB are double of the same angle CAB, or CAD is equal to CDA; wherefore (I. 5. El.) CD, being equal to AC, is given. Again (I. 30. El.) the angle ACD is equal to the angles ABC and BAC or CAD, and hence the angle ABC, which is given, is equal to the difference between the angles ACD and CAD. The triangle ADC has therefore its base AC given, and the difference between the adjacent angles; whence the *locus* of the vertex D is a given rectangular hyperbola. But CD being also given, the point D lies in the circumference of a given circle; therefore D is given, and consequently DCB, and the point B of trisection.

COMPOSITION.

With conjugate diameters equal to the chord AC, and inclined at the angle ABC contained in the given segment, construct an equilateral hyperbola, from C with the radius CA describe a circle intersecting the curve in D, and draw DCB; the arc ABC is trisected in B.

For CA being equal to CD, the angle CAD is equal to CDA; but, from the construction, the angle ABC is equal to the difference of the angles ACD and CAD; that is, the angle ACD, which is equal (I. 30. El.) to the angles ABC and CAB, is equal to the angles ABC and CAD; whence the angle CAB is equal to CAD. Again (I. 30. El.) the angle ACB is equal to CAD and CDA, or the double of CAB; whence the arc AB is double of BC, or BC is the third part of the whole arc ABC.

PROP. XXVII. PROB.

The base and the difference of its adjacent angles being given, to find the locus of the vertex of the triangle.

Let it be required to find the locus of C (fig. 155.), the vertex of the triangle ABC, of which the base AB and the excess of the angle ABC above BAC are given.

ANALYSIS.

Draw CD, making an angle BCD equal to BAC, and meeting the extension of AB in D. The angle ABC being equal (I. 30. El.) to the angles BDC and BCD or BAC, the angle BDC is consequently equal to the difference of the angles ABC and BAC at the base, and therefore the line CD has a given inclination to AD. But the triangles ACD and CBD, having the same vertical angle at D, and the angle DAC, by construction, equal to DCB, are (VI. 11. El.) similar; whence AD : DC :: DC : DB, and (V. 6. El.) $AD.DB = DC^2$. Wherefore the vertex C lies in a given rectangular hyperbola, of which AB and an equal straight line parallel to DC are conjugate diameters.

COMPOSITION.

Bisect AB in O, and through this middle point draw IOK equal on both sides to the portions OA or OB, and making an angle IOA equal to the difference of the angles BAC and ABC at the base, and describe a rectangular hyperbola, of which O is the centre, and AB, IK conjugate diameters. This curve is the locus required.

For, from any point in it as C, and parallel to IK, draw CD meeting the conjugate diameter AB produced in D. From the property of this curve $AD.DB = DC^2$, and therefore (V. 6. El.) AD : DC : : DC : DB; whence the triangles ACD and CBD, having likewise a common vertical angle at D, are (VI. 13. El.) similar, and the angle BCD is equal to BAC. But (I. 30. El.) the angle ABC is equal to BDC and BCD or BAC, and consequently ADC or (I. 22. El.) the given angle AOI is equal to the difference of the angles ABC and BAC.

The construction may be simplified, by drawing EBF at angles equal to the difference of those of the base, and having made BE and BF equal to OB, extend the perpendicular asymptotes OE and OF.

Scholium. If the *sum*, instead of the *difference* of the angles at the base of the triangle were given, the locus of the vertex would become a *circle*, which is indeed only the inversion of the *rectangular hyperbola*. For all the angles of the triangle ABC being equal to two right angles, and the sum of the angles at the base being given by hypothesis, the vertical angle ACB is hence given, and consequently (III. 16. El.) the vertex C must lie in a given arc of a circle.

But the analysis may be conducted in the same way as when the *difference* of the angles at the base was given. Draw CD′ making an angle BCD′ equal to BAC, and produce it till BC′ be equal to BC; then AD′C, being equal to D′BC and BCD′ or BAC, is equal to the *sum* of the angles at the base, and therefore given. Now the triangles AD′C and BD′C′ are evidently similar, and consequently AD′ : D′C : : DC′ : D′B, and (V. 6. El.) AD′.D′B=DC.DC′. But this is the property of a circle passing through the two points A and B, or having its diameter bisecting AB at right angles; and since the chord BC is equal to BC′, another diameter must cut CC′ at right angles, and is consequently given in position, whence the circle is given.

If the *sum* of the angles at the base be a right angle, the vertical angle at C must also be a right angle, and hence contained in a semicircle. Whence the *difference* of the angles at the base becomes a right angle, the conjugate diameters AB and IK pass into the transverse and conjugate axes.

PROP. XXVIII. PROB.

To find a point in the circumference of a given circle, from which straight lines drawn to two given points shall be equally inclined on either side of the radius.

Let D (fig. 156.) be the centre of the given circle, and A, B two given points; it is required to find a point C in the circumference, such that CA, CD and CB being drawn, the angle DCA shall be equal to DCB.

ANALYSIS.

Join DA and DB; from C draw CE and CF, making the angles DCE and DCF equal to DAC and DBC, and meeting DA and DB produced in E and F, and join CE, CF and EF. The triangles CDA and EDC, having the same vertical angle, and the angle DAC by construction equal to DCE, are similar. In like manner, the triangles CDB and FDC are proved to be similar. Wherefore AD : DC : : DC : DE, and consequently AD and DC being given, the third proportional DE is given, and therefore the point E. Again, DB : DC : : DC : DF, whence DF and the point F are given. But the triangles CDA and CDB being similar to EDC and FDC, the angles DEC and DFC are equal to DCA and DCB, which are equal by hypothesis; whence adding DEF and DFE to each, the difference between the angles CEF and CFE must be the same as the difference between the angles DEF and DFE. Now, the triangle EDF being given, the difference between the angles DEF and DFE is given; whence the triangle ECF having its base EF given, and the difference between the adjacent angles CEF and CFE, its vertex C must lie in a given rectangular hyperbola, by the last Proposition.

COMPOSITION.

Join DA, DB and AB, find (VI. 3. El.) a third proportional to DA and the radius, and to DB and the radius, and produce DA and DB till DE and DF be equal to those lines; join EF, and bisect it in O; through O draw GOH parallel to AB, and having its portions OG and OH equal to EO or OF, and construct a rectangular hyperbola to the conjugate diameters EF and GH; the intersection of this curve with the circle will assign the required point C.

For, since by construction AD : DC : : DC : DE, and BD : DC : : DC : DF, the rectangles AD.DE and BD.DF are both equal to the square of DC; whence AD.DE = BD.DF, and AD : BD : : DF : DE; wherefore the triangles ADB and FDE, having a common vertical angle, are similar, and consequently the angle DAB equal to DFE; but DAB (I. 22. El.) is equal to AIO, which again (I. 30. El.) is equal to DEF and EOI, and hence GH makes with EF an angle equal to the difference of the angles DEF and DFE. Now, the point C lying in a rectangular hyperbola, of which EF and GH are conjugate diameters, the angle EOG must, by the last Proposition, be equal to the difference between the angles CEF and CFE, and therefore the angles CED and CFD on both sides must be equal. Again, because AD : DC : : DC : DE and BD : DC : : DC : DF, the triangles ADC and BDC are similar; whence the angles CED and CFD are equal to ACD and BCD, which are therefore equal.

PROP. XXIX. PROB.

The segment of a curve of the second order being given, to find either absolutely or approximately its area.

1. *In the Parabola.*

To find the space ABC (fig. 71.) included by a parabolic arc, and its diameter and ordinate.

Conceive the absciss AC to be divided into minute intervals E*e*, at which the ordinates ED, *ed*, &c. are applied, and from their extremities drawing parallels D*m*, &c. to the diameter AC, the curved space will be occupied by a polygon consisting of elementary rhomboids ED*me*; join the contiguous vertices D, *d*, by the straight line HD*d*; complete the rhomboids HIDE, H*ide*, and apply the vertical tangent AG. The complements D*e* and D*i* about the diagonal of the rhomboid A*edg* are (II. 8. El.) equivalent; but the nearer that the point D is taken to *d*, or the secant HD*d* made to approach the position of a tangent at D, the nearer is HE the double of A*e*, and the complementary rhomboid D*i* hence approximates to the double of Dg. Wherefore the element D*e* is ultimately double of D*g*; and consequently the aggregate interior polygon, filling up the curved segment, is double of the exterior polygon, which completes the circumscribing rhomboid AEDG. Whence the space ADBCE is the double of AGKBD, and consequently the parabolic segment is two-thirds of the circumscribing rhomboid.

2. *In the Ellipse.*

It is required to approximate to the area of the curve ABC (fig. 74.) included by the elliptical arc and its diameter and ordinate.

Conceive the abscis AC to be divided into minute sections E*e*, the adjacent ordinates ED, *ed* applied, and the elementary rhomboids ED*me*, and E*nde* formed; let the diameter AL describe the semicircle AHL, erect the perpendiculars CH, *eg* and EG, and complete the rectangles GE*ge*, E*rge*; and, lastly, in the ellipse draw the semiconjugate diameter OI, and let fall upon AL the perpendicular IK. The elementary rectangle EG*ge* is to the elementary rhomboid ED*me*, which has the same base, as the altitude EG to the perpendicular from D upon AL, or in the ratio compounded of EG to ED, and of ED to that perpendicular. But, from the property of the ellipse, EG is to ED as OA to OI; and, from similar triangles, ED*me* is to a perpendicular from D as OI to IK; wherefore, the minute rectangle EG*ge* is to the minute rhomboid ED*me* as OI to IK; and, for a like reason, E*rge* is to E*nde* in the same ratio. Now the rectangles EG*ge* and E*rge* are elements of the polygons inscribing and circumscribing the circular segment AGE, and the rhomboids EDme and Ende are the elements of the elliptical segment ADE; whence the former is to the latter, as OI to IK, and therefore the circular segment itself is to the corresponding elliptical one ultimately in that ratio.

3. *In the Hyperbola.*

Let it be required to approximate to the hyperbolic space ZMHGA, (fig. 77.) included by an arc of the curve and the asymptote. Find (VI. 18. El.) OB a mean proportional between OZ and OA; draw the parallel BH, and make *zb*=2ZB. The space ZMHB is (Prop. 23.) cor. 1.) equivalent to BHGA, or the whole hyperbolic space ZMHGA is double of ZMHB; and consequently the rhomboid contained by the double of ZB or by *zb* and ZM, and under the angle ZOA of the asymptotes, is greater, and the like rhomboid, contained by *zb* and BH is less than the whole space. Again, find OC, a mean proportional between OZ and OB, draw the parallel CI, and make *zc*=4ZC. The spaces ZMIC and CIHB are equivalent, and each of them is hence the fourth of the whole hyperbolic space ZMGA, which must therefore be greater than the rhomboid under *zc* and CI, and less than that under *zc* and ZM. In the same manner, OD and OE, &c. being found successive mean proportionals to OZ, and OC, OD, &c. and having made *zd*=8ZD, *ze*=16ZE, &c. the area of the hyperbolic space will be confined within the approximating limits of the rhomboid contained by *ze* and ZM, and one contained by *ze* and EL. Hence OX is to XY, as the rectangle under ZM and the ultimate length of *ze*, to the measure of the hyperbolic space ZMHGA.

Cor. The quadrature of the parabola is hence always assigned geometrically, the quadrature of the ellipse may be derived from that of the circle, and the quadrature of the hyperbola can be expressed numerically by help of a table of logarithms.

PROP. XXX. PROB.

To find the length of a given portion of the curve of a given parabola.

Let it be required to find the length of an arc AC of (fig. 156.) the parabola CAM.

ANALYSIS.

Apply the vertical tangent EAR; near C assume any point *c*, and draw CE and *ce* parallel to the axis; erect CH perpendicular to the chord C*c*, and draw CG and *ck* parallel to the vertical tangent. The triangles HGC and *ck*C are evidently similar, and consequently CH : GH : : C*c* : *ck*; whence CH.*ck* = GH.C*c*. Let the point *c* approximate to C, and GH will become the subnormal, and equal to half the parameter of the axis. In HA produced, make AB equal to this semiparameter, join BE, and produce CE till ED be equal to it. Then CH.*ck* = ED.E*e*, and therefore AB.C*c* = the space ED*de*, when *ed* approximates to ED. Consequently, collecting the component portions ABC*c*, and the corresponding parallelogramic spaces ED*de*, it follows that the rectangle under AB, and a straight line equal to the arc CO is equivalent to the curvilineal space EDKP. But since ED is equal to the oblique line EB, the curve DKB is a rectangular hyperbola, which has AB for its semitransverse axis. Now, the area of this hyperbola being given by the last proposition, the exterior space EDKP is also given, and there fore the rectangle under CO and the semiparameter AB; whence CO itself is given.

But the difference between two given related arcs of the parabola is more easily determined. For, draw any parallel chords DI and KL, and likewise IRM and LQN parallel to the axis. From what was before shown, it follows that $AB.MN =$ curvilineal space QLIR, and therefore, that $AB\,(MN - CO) = QLIR - EDKP$. But KD and LI being joined, the narrow spaces included between those chords and the curve must be equal. Wherefore the difference between the curvilineal spaces EDKP and QLIR is equivalent to the difference between the quadrilateral figures PEDK and QRIL. But the areas of these trapezoids are the rectangles under their bases, and the sum of their altitudes. Whence, $AB(MN - CO) = \frac{1}{2}QR\,(QL + RI) - \frac{1}{2}EP(DE + PK)$.

If L coalesce with B, the chord KL will pass into BS, a parallel to DI. Wherefore $AB\,(AM - CO) = \frac{1}{2}AR\,(AB + RI) - \frac{1}{2}EP(DE + SV)$.

Scholium. The *Rectification of the Parabola* thus depends on the quadrature of the hyperbola combined with that of a rectilineal figure, and the length of any portion of the curve is therefore expressed by a logarithm, joined to a common algebraic quantity. It is not difficult to show that the rectangle under the arc AC and the semiparameter AB, which is equivalent to the mixtilineal space ABDE, might be denoted by $\frac{AB}{2}.\, log.\, \frac{BE + AE}{AB} + \frac{(BE - AE)\,AE}{2}$.

It might likewise be proved, that if AV were taken equal to half the focal distance, or the fourth part of the semiparameter AB, of the parabola, and an equilateral hyperbola constructed, having the points V and A for its centre and vertex, or assuming VX equal to AV, and drawing BY to the curve; then will the rectangle under AV and the parabolic arc AC be equivalent to double the space XYA compounded of the triangle XYV and of the hyperbolic sector VYA.

An approximation may be easily obtained in the case of small arcs; for considering the incurvation of BD as merely parabolic, the area of the mixtilineal space ABDE will be AE $\left(\frac{2\,AB+DE}{3}\right)$. But DE or BE is nearly $=$ $AB+\frac{AE^2}{2\,AB} = BG$, and consequently the space ABDE$=$ $\frac{CG}{3}\,(3\,AB+AG)$.

This result corresponds with the Scholium to Prop. 6. of the Trigonometry, the parabolic arc AC nearly coinciding with the osculating circle of which AB is the radius.

PROP. XXXI. THEOR.

Parabolas are all similar figures.

Let BAC and *bac* (fig. 157.) be two parabolas, whose focal distances are AF and *af*; they are similar figures, all their parts being constructed after the same proportion.

For make AC : *ac* : : AF : *af*, divide AC and *ac* into any number of equal parts, and from the points of section draw ordinates to the curves. Since AF : *af* : : AD : *ad*, therefore $AF^2 : af^2 :: 4AF.AD : 4af.ad$; but, from the property of the parabola, $DG^2=4AF.AD$, and $dg^2= 4af.dg$; whence $AF^2 : af^2 :: DG^2 : dg^2$, and AF : *af* : :

DG : *dg*. In like manner it is proved, that all the other ordinates are in the ratio of AF to *af*, or of the mutual interval AD to *ad*. The inscribed polygons AGHBC and *aghbc* are therefore similar; but this property must obtain whatever be the number of sides, and consequently their extreme limits, or the semi-parabolas themselves, must be similar. The entire parabolas are hence also similar.

The circle, the parabola, and the equilateral hyperbola, being only particular shoots from the general curve, retain each of them its unvaried feature, while the ellipse and oblique hyperbola exhibit great diversity of forms.

BOOK III.

GEOMETRY OF CURVES.

I. CISSOID.

If from the extremity of the diameter of a circle any straight line be drawn to meet a tangent applied at the other extremity, and a portion be cut from it equal to the intercepted chord, the locus *of the point of section is a Cissoid.*

Extend any straight line CR (fig. 158.) from C, the end of
the diameter CA, to meet AR, the tangent applied at the
F other end, and make ER equal to the chord CF; the
F point E will trace a *Cissoid*.

If the perpendicular diameter OD be drawn, it is obvious, that the intersection D will bisect the segment FE intercepted between the curve and the circumference of the circle. For the same reason, the perpendiculars FH and EI must lie at equal distances OH and OI from the centre, and the extreme segments CH and AI must likewise be equal.

The revolving line CR may be conceived to turn on both sides of the diameter CA, till it vanishes away in the position of a tangent to the circle at C. The Cissoid, therefore, consists of two branches, which issue from the point C, pass through G and N, the extremities of the perpendicular diameter, and stretching upwards and downwards, approaches always nearer to the tangent RS. But since CH is equal to IA or EQ, the curve, notwithstanding its continual approximation, must yet be separated from RS by some definite interval.

DEFINITIONS.

1. The circle CGAN is called the *Generating Circle.*
2. The point of origin C is named the *Cusp of the Cissoid.*
3. The primary diameter CA is termed the *Axis of the Cissoid.*
4. The extended tangent RS is called the *Directrix* or *Asymptote of the Cissoid.*
5. Any segment CI of the axis intercepted from the cusp, is termed an *Absciss,* the perpendicular IE to the curve being named an *Ordinate.*

PROP. I. THEOR.

If, through any point in the Cissoid, a straight line be drawn parallel to the line joining the corresponding point in the circumference of the generating circle with the centre and meeting the perpendicular diameter, it will be equal to the radius; and if it be extended to the same distance on the other side, and a perpendicular erected, this will cross the prolongation of the axis at a point situate beyond the cusp at the interval likewise of the radius.

Let the revolving line CR (fig. 159.) cut the generating circle and the Cissoid in points F and E, join OF, and parallel to it draw EL, which will be shown to be equal to OF; produce LE till EM be likewise equal to the radius, and erect the perpendicular MK to meet the extended axis in the point K, between which and the centre O the cusp C of the curve will be equidistant.

For join KL and OM. Since EL is parallel to OF, the angles DOF and DFO are equal to the alternate angles DLE and DEL; and, from the genesis of the curve, the corresponding sides DF and DE are equal; whence the triangles ODF and LDE are equal, and therefore the side OF is equal to LE, and OD to DL. Again, EM being equal and parallel to OF, the figure OFEM must be a parallelogram; and since the transverse lines OM, CE and KL divide the converging lines KO and LM into equal segments, they must all be parallel; and hence the angle LKO is equal to FCO, and KLM is equal to CEM or CFO. But the triangle COF being isosceles, the angle FCO is equal to CFO, and consequently the angle LKO is equal to KLM. Wherefore the right-angled triangles LKM and LKO, having likewise the acute angles LKO and KLM equal, and the side KL common, are equal; the side OK is hence equal to LM, or to the diameter AC. Wherefore the exterior segment CK is equal to the radius OC.

Scholium. Hence the organic description of the Cissoid is easily derived; for, in the extension of the axis, having assumed CK equal to CO, let a right angle KML, whose side LM is equal to the diameter and bisected in E, and the other side KM is of indefinite length, travel through the point K, while the extremity L slides along the perpendicular diameter OD; then will the middle point E trace out the curve.

PROP. II. THEOR.

Any absciss and ordinate of the Cissoid, and the corresponding absciss and ordinate of the generating circle, are in continued proportion.

Let the revolving line CR (fig. 158.) cut the circle and Cissoid in the points E and F, from which let fall the perpendiculars EI and FH; then CI : EI : : EI : AI or CH : : CH : FH.

For EA being joined, the angle CEA in the semicircle is a right angle, and FH is a mean proportional between the segments CI and IA or CH. But, from the property of parallel and diverging lines, CI : EI : : CH : FH; wherefore, collectively, CI : EI : : EI : CH : : CH : FH.

PROP. III. PROB.

At a given point in the Cissoid to apply a tangent.

Let it be required to draw a straight line (fig. 160.) that shall touch the curve at the point E.

Adjacent to the radiant CR, assume another radiant C*r*, draw the secant CETX to meet IX a parallel to CF, and from C describe the minute arcs *f*φ and E*s*. It is evident, that the elementary arc F*f* being viewed as a portion of a tangent to the circle at F, and hence the angle CF*f* being equal to CAF or CEI, the elementary triangle Fφ*f* is similar to FCA, and the elementary or nascent triangle FC*f* is similar to *s*CE. But since, from the nature of the curve, CF is equal to ER, and C*f* to *e*R, and that Cφ is equal to εR, it follows, that φ*f* is equal to *e*ε; wherefore the ratio of Eε to ε*e*, is compounded of the ratios of Eε to F*f* and of F*f* to φ*f*, that is, compounded of the ratio of CE to CF, or EI to FH, and of the ratio of CF to FH or of FH to the parallel HS. Consequently, Eε : ε*e* : : EI : HS; but, from the property of parallel and diverging lines, Eε : ε*e* : : EI : IX; and hence EI : HS : : EI : IX, and IX is therefore equal and parallel to HS, when the points Eε and *e* collapse and the secant merges into a tangent.

An easy construction is thus obtained. Draw HS parallel to CF, and meeting the chord AF, join IS, and complete the parallelogram HSTX; the oblique line EX will be the tangent required.

Cor. It is plain, that in tracing the progress of the curve from its cusp to its indefinite extension along the asymptote, the intersection T of the tangent likewise travels through the diameter from C to A. The curve is hence always convex towards the diameter.

PROP. IV. PROB.

To find two mean proportionals between two given lines.

Let it be required to find two mean proportionals (fig. 158.) to the lines OA and OT.

Place the longer line OA at right angles to the other, and with it, as a radius, describe a circle, to which trace the corresponding Cissoid, and draw the diagonal AT to cut the curve in F, join CF; and the segment OD, which this line intercepts on the perpendicular diameter, will be the first of the two mean proportionals to OA and OT.

For CI, EI, CH and FH were proved to be in continued proportion, and they must evidently be still proportionals when diminished in a given ratio. But CI : CO : : EI : OD, and CI or AH : CO or AO : : FH : OT.

To find the other mean proportional, draw DS parallel to CA, and join CS cutting OD in X; then CI : CO : : OD or IS : OX.

PROP. V. THEOR.

The space lying between the double branches of the Cissoid and its asymptote is equivalent to triple the area of the generating circle.

The space comprehended between the Cissoid as it stretches on both sides of the diameter AC (fig. 161.), and bends indefinitely towards the directrix or asymptote RS, is three times the area of the generating circle.

For let CFER be any revolving line, draw another Cfer indefinitely near it, and complete the parallelogram ERrs.

Because the square of the hypotenuse CR is equivalent to the squares of AC and AR, and the rectangle CR, RF is itself equivalent to the square of AR, the remaining rectangle CR, CF must be equivalent to the square of AC, which, for the same reason, is equivalent to the rectangle Cr, Cf, wherefore CR : Cr : : Cf : CF; and conceiving the small arc Ff as coinciding with a tangent at F, the triangles FCf and RCr are similar, and CF : CR : : Ff : Rr; but CF : CR : : CH : CA, and consequently, CH : CA : : Ff : Rr, and the rectangle under CA and Ff is equivalent to the rectangle under CH or EQ and Rr. Now the rectangle under the diameter CA and Ff is evidently equivalent to four times the elementary triangle FOf; and the rectangle under the perpendicular EQ and the base Rr is the area of the parallelogram ERrs. Again, since CF is equal to ER or sr, and the alternate angle FCf is equal to sre, the elementary triangles CFf and rse are equal. Wherefore the elementary space ERre is equivalent to the excess of four times FOf above FCf; and hence the aggregate space ACGER, lying above the diameter AC and bounded by the curve, the line ER, and the asymptote AR, is equivalent to the excess of four times the sector AOF above the compound segment AGFC, or the excess of the triangle AOF, and the segment AGF quadrupled, above double the triangle AOF with the segment AGF.

Cor. Hence the quadrantal crescents CG and CN formed by the Cissoid are equivalent to the space included between the circle and its circumscribing square. For the mixtilineal space ACG is equivalent to double the triangle AOG or ACG and thrice the segment AG, take ACG from both, and the triangle AGV or AGC is equivalent to thrice the circular segment AG with the segment CG of the Cissoid; wherefore the inscribed square, or the difference between it and the circumscribing square, is equivalent to the two crescents CG and CN, with the four segments AG; GC, CN and AN; but these segments, with the interior square, make up the circle, and consequently the crescents are equivalent to the space between the circle and its circumscribing square, that is, equivalent to thrice the segment AGF and double the triangle AOF. Consequently, when the revolving line CFR vanishes into a tangent YCX, the triangle AOF becomes extinguished, and the curved asymptotic space above the diameter AC is equivalent merely to three semicircles; but the corresponding space below the diameter having the same measure, the joint space between the curve and its extended asymptote must be equivalent to three times the generating circle.

The *Cissoid* of Diocles was one of the first curves produced in the Platonic School, after the discovery of the Conic Sections or Lines of the Second Order. The object in view was, to solve the famous problem of the *Duplication of the Cube*, or the *Insertion of two mean proportionals between two given lines.* The *Cissoid* received its name from κισσος, the Greek word for *Ivy*, because the curve appears to mount along its asymptote in the same manner as that parasite plant climbs on the tall trunk of the pine.

Newton himself first gave, in his *Universal Arithmetic*, the mechanical description of the Cissoid, by the motion of a rectangular ruler, and proposed this curve as a convenient medium for the construction of the Higher Equations.

II. CONCHOID.

If, through a given point, a straight line be drawn to a line given in position, making on either side a given segment; the locus *of the point of section is a* CONCHOID.

Let P (fig. 162, 163, and 164.) be a given point, through which any straight line is drawn, to cut or meet another straight line AD given in position; if a given segment AB be taken on either side, the point B will trace a *Conchoid.*

DEFINITIONS.

1. The given point P is called the *Pole of the Conchoid.*
2. The given line AD is named the *Directrix,* the perpendicular PDE the *Axis,* the part DE the *Diameter,* and the point D where this stands on the directrix, is called the *Centre* of the curve.
3. Any perpendicular CB to the directrix is termed an *Ordinate,* to which the intercepted portion CD from the centre is the *Absciss.*
4. When the section B of the radiating line PA lies beyond the directrix, (fig. 162.) the curve is termed an *Exterior Conchoid;* if that point lies between the pole and the directrix, (fig. 163.) the curve is termed an *Interior Conchoid;* but when the segment AB exceeds PD, (fig. 164.) and the point B, during part of its tract winds, beyond the pole, the curve is termed a *Nodated Conchoid.*

PROP. I. THEOR.

An ordinate to the Conchoid is the cosine of the inclination of the radiant to a radius equal to the diameter, and the absciss is the sine of that angle to a radius equal to the radiant itself.

It is evident, that (fig. 162, 163, and 164.) AB, which is equal to the diameter DE, being made a radius, the ordinate BC will be the cosine of the angle ABC or DPA the inclination of PAB to the axis of the curve. It is likewise apparent, that the absciss CD or BF is the sine of the same angle DPA to the compound radius PB.

Cor. Let a denote the axis PD, d the diameter DE, and i the angle DPA; then $BC = d.\cos i.$ $CD = d.\sin i + a.\frac{\sin i}{\cos i}$ $= d.\sin i + a.\tan i.$ Whence BC continually diminishes as CD augments, and the directrix must therefore be an asymptote to the conchoid.

PROP. II. PROB.

To apply a tangent at a given point of the Conchoid.

Let it be required (fig. 162, 163, and 164.) through the point B to draw a tangent BT to the curve.

Join BP with the pole, and let another radiant bP approach it, extend the secant Bb to meet the directrix in T; from A draw Aβ equal and parallel to ab, join $b\beta$ and Bβ, which produce to G.

Since Aβ is by construction equal and parallel to ab, the figure $ab\beta$A is a parallelogram, and its opposite sides Aa

and β*b* are likewise equal and parallel. It is hence obvious that the triangle BGT is similar to the elementary triangle Bβ*b*, and BG : GT : : Bβ : B*b*. But the vanishing chord Bβ of a circle, of which AB and Aβ are radii, may be considered as perpendicular to AB, or parallel to a corresponding perpendicular Aα. The ratio of Bβ to β*b* or A*a* is compounded of the ratio of Bβ to Aα and of Aα to A*a*; that is, of the ratio of AB to AP, and of BC to AB. Wherefore Bβ : β*b* : : BC : AP, and hence BG : GT : : BC : AP; but, from similar triangles, AG : BG : : AB : BC, and (V. 16. El.) AG : GT : : AB : AP, or, by composition, AG : AT : : AB : BP.

Whence an easy construction is derived. For draw BG perpendicular to the radiant PB, and produce it to meet a parallel from the pole to the directrix in H; and AT being made equal to PH, or HT being drawn parallel to AP, will assign the point T where the tangent terminates.

Cor. It is evident that the tangent at the vertex E will be parallel to the directrix. It also appears that, in the extreme verge, the tangents will vanish into parallelism. But in the intermediate distances, the subtangent DT contracts within moderate limits. Consequently there must be a certain position on either side, at which the subtangent is the least possible, and the curve suffers a contrary inflection.

The same observations apply to the interior conchoid. But in the case of the nodated conchoid, the construction becomes indeterminate at the pole P, where the opposite branches of the curve cross each other. To find the tangents at that point, inflect towards the directrix the two lines P*t* and *pt* equal to the axis.

PROP. III. PROB.

To trisect a given angle by the application of the Conchoid.

Let it be required, by help of this curve, to divide the angle ABC (fig. 177.) into three equal angles.

From B as a centre, with any radius BA, describe a circle; supposing the angle ABD to be the third part of ABC, draw DF parallel to AB, join CF and produce it to meet the extension of AB in G. In the 31st Proposition of the First Book of the Geometrical Analysis, it was demonstrated, that the intercepted segment FG is equal to the radius of the circle; whence the point C may be the pole of a Conchoid, of which AG is the directrix, CG a radiant, and FG the diameter of curve.

Having therefore constructed an exterior and interior, or nodated Conchoid to the pole C, the directrix AG, and the diameter AB; let it cut the circumference of the circle in the points F, F′, and F″. Join BF, BF′, and BF″, and draw the parallel radii BD, BD′, and BD″, which will mark the triple section of the angle ABC.

It may be perceived that the exterior branch of the Conchoid intersects the lower semicircle in another point besides F″. This occurs in the extension of CB, or where the diameter passing through C, the extremity of the original arc, meets the opposite circumference; the portion of the inflected line intercepted below the directrix AG being evidently equal to the radius. This fourth intersection, however, affords no actual solution, but only exhibits the process of repeated division as completing the angle itself.

Another mode of construction.

But the second analysis, which was given of the same Problem, affords a different solution. Let it be required to trisect the angle ABC (fig. 178.) draw AC perpendicular and CF parallel to AB; it has been demonstrated that if the segment DF of the inflected line BF were equal to the double of BC, the angle ABD would be the third part of ABC; whence the construction is reduced to the describing of a Conchoid from the pole B to the directrix AC, with the double of BC as its diameter; the intersection of the curve with the perpendicular CF will determine the position of the trisecting radiant BDF. The exterior branch of the Conchoid will cut this perpendicular in the point F, and the interior or nodated branch will meet and cross it at the two points F′ and F″. The radiating lines BF, BF′ and BF″, or its extension B*f*, will complete the trisection of the angle ABC. These lines will be found, as in the first construction, to make angles with each other that are equal to the thirds of an entire circuit.

It may be worth while to examine the successive cases. In the second position DBF′ of the trisecting line, draw CG′ to the middle point. Because the triangle F′G′C is isosceles, its exterior angle CG′B is double of CF′G′ or of the angle ABD′; but G′CB being also an isosceles triangle, CG′B is equal to CBG, and is consequently double of ABD. Add CBD, which is the double of ABD, and the compound angle DBF′ is double of DBD′, which would complete two right angles; whence DBF′ is two-thirds of two right angles, or one-third of a whole circuit.

In the third position BF″D″ of the trisecting line, or rather its extension A*f*″, draw CG″ to bisect F″D″. The triangles CG″D and BCG″ are their isosceles, and consequently the angle CBD″ or CG″B is double of CD″B; but CBF is likewise double of CFB, and therefore the combined angle D″GF is double of the angles CD″B and

CFB. Now this angle D″GF, together with the two angles CD″B and CFG, is evidently equal to the exterior angle DC″F or a right angle. Whence D″BF is two-thirds of a right angle, or one-third of two right angles, and therefore the adjacent angle DBF″ is two-thirds of two right angles, or one-third of a whole circuit.

PROP. IV. PROB.

By the application of the Conchoid to find two mean proportionals between two given lines.

Let it be required (fig. 179.) to find two mean proportionals to the perpendiculars AB and AC, by help of this curve.

In Book I. Prop. 32. of the Geometrical Analysis, it was shown, that, having completed the rectangle BACD, bisected BD in F, constructed upon the base AB an isosceles triangle ABK with its sides AK and BK equal to BF, produced AB till BL be equal to it, joined KL, and drawn AM parallel to this; if, from the vertex K of the isosceles triangle, the straight line KMG were inflected such, that the part MG intercepted between AM and the extension of BA shall be equal to BF the half of AC, the oblique line GCE would cut off the segments DE and AG, which are mean proportionals to AB and AC. Wherefore describe an Exterior Conchoid, of which K is the pole, AM its directrix, and BF its diameter; and the intersection of the curve with the line BA produced will assign AG the second of the two mean proportionals, and GC being drawn to meet the extension of BD, will indicate DE the first term.

The *Conchoid* had its name from κονχος, *a shell;* being invented by Nicomedes for the *Duplication of the Cube* and the *Trisection of an Angle.*

III. QUADRATRIX.

If a straight line turn uniformly about a given point, while another straight line advances uniformly in a parallel position, their mutual intersection will describe the curve called a Q*UADRATRIX*.

Let the coincident straight lines OI and IOH, (fig. 166.) begin their motion, the former turning uniformly about the point O, and the latter advancing uniformly parallel to its first position; the intersection B resulting from this combined revolving and progressive movement will trace out the *Quadratrix.*

Suppose, while the revolving line describes a right angle, the parallel advances to E; this must be a point in the curve. But in describing the second right angle, the revolving line will stretch out, and will itself become parallel to the position FG, which the progressive parallel attains when it has passed over a space EF equal to OE. At this limit, the curve must vanish into distance. In the description of the third right angle, the intersection will begin again beyond H, will travel through E at an interval beyond F, equal to OE, and will shade away towards G along a second asymptote placed at an equal distance beyond the first. A similar recurrence will take place at every succeeding two right angles; and thus the quadratrix must consist of a series of innumerable branches, extending indefinitely both to the right and to the left.

Since the revolving and progressive motions involved in the genesis of this curve are uniform, it follows, that the spaces which they describe will be proportional. Wherefore OE is to the perpendicular BD, as the right angle EOI to the oblique angle BOI; or, if a circle be described from the given point O, and passing through E, the radius OE will be to BD as the quadrant IPE is to the arc IP.

DEFINITIONS.

1. The intercepted portion OA of the revolving line in its first position is called the *Base* of the curve, and the extended perpendicular EFF′F″ &c. its *Axis*.
2. A perpendicular BC to the axis is named an *Ordinate*, and its distance CO or BD from the centre of revolution its *Abscis*.

PROP. I. THEOR.

The base of the Quadratrix is a third proportional to the quadrant and its describing radiant.

Let IE (fig. 167.) be the quadrant traced in the evolution of a right angle, the radius being OE; then will IE : OE : : OE : OA.

For suppose O*ai* to be the position of the revolving line, the instant after it has commenced its motion. The elementary arc A*a* may be viewed as coinciding with a perpendicular to the base, and therefore IE : I*i* : : OE or OI : A*a*, or alternately, IE : OI : : I*i* : A*a*; but I*i* : A*a* : :OI : OA, and consequently IE : OI : : OI : OA.

Cor. 1. Hence OA : 2OI : : 2OI : 4IE, or OA : IH : :

IH : 4IE; that is, the base OA is to the diameter IH, as this diameter to the circumference of its circle.

Cor. 2. Hence likewise the square of the diameter is to the area of the circle, as the base OA to half the radius OI; for, by composition, IE.2OI or IE.IOH : $4OI^2$ or IH^2 : : 2OI or IH : OA; but IE.IOH is the area of the circle, and the ratio of IH to OA is the same as that of OI to the half of OA. The description of this curve would thus give the quadrature of the circle, and hence its name *Quadratrix.*

Cor. 3. Since BD : PI : : OA : OI, and BD : PQ : : OB : OP or OI; it follows, that PQ : PI : : OA : OB. Wherefore the sine of the angle generated is to its corresponding arc as the base of the curve is to the radiant.

PROP. II. THEOR.

The base of the Quadratrix being made the radius of a circle, the arc of revolution will be equal to the corresponding absciss of the curve.

If from the centre O, (fig. 167.) with the radius OA, a circle be described, the arc AR intercepted by a radiant is equal to BD the absciss to the curve.

For the absciss BD is to the arc IP as OA to OI, and consequently as the arc AR to the arc IP; wherefore the elapsed arc AR must be equal to BD.

Scholium. If OK be made equal to any radiant OB, and BK joined, the bisection of this base of the isosceles triangle BOK will assign a point L in the Quadratrix. For the perpendicular LM is evidently the half of BD, while the radiant OL bisects the angle IOP; the point L would hence be attained by just half the progressive and revolving movements which determine B. Repeating this process, a succession of points may be found approaching always nearer to A, and the arc AB thus determined would be measured by the assumed straight line BD.

An approximation to the quadrature of the circle could in this way be discovered mechanically with great facility and tolerable precision. Let OA (fig. 165.) be a radiant corresponding to a right angle, the lines OB, OC, OD, OE, &c. drawn from the centre O to bisect the bases of the successive isosceles triangles, will exhibit the radiants of the series of bisected angles; the final radiant OZ, which is the base of the Quadratrix, being made the radius of a circle OA, will be equal to its quadrant. The same construction will apply when the radiant of an oblique angle is taken; but, if the assumed angle should exceed a right angle, the first of the isosceles triangles will likewise be obtuse.

This curious property may be viewed as a modification of Prop. 6. of the Trigonometry and its corollaries. If r denote the radiant of any angle φ, the radiants corresponding to $\frac{1}{2}\varphi$, $\frac{1}{4}\varphi$, $\frac{1}{8}\varphi$, &c. will be expressed by $r.\cos\frac{1}{2}\varphi$, $r.\cos\frac{1}{2}\varphi.\cos\frac{1}{4}\varphi$, $r.\cos\frac{1}{2}\varphi.\cos\frac{1}{4}\varphi.\cos\frac{1}{8}\varphi$; and the radius of the circle which has r for the length of the arc that measures the angle φ will be

$$r.\cos\tfrac{1}{2}\varphi.\cos\tfrac{1}{4}\varphi.\cos\tfrac{1}{8}\varphi.\cos\tfrac{1}{16}\varphi.\cos.\tfrac{1}{32}\varphi.\ \&c.$$

In the case selected for illustration, OZ = OA . $\cos 45°.\cos 22\frac{1}{2}.\cos 11\frac{1}{4}.\cos 5°\frac{5}{8}.\cos 2°\frac{13}{16}$ &c.

PROP. III. THEOR.

If, from the extremity of the base of the Quadratrix, straight lines be drawn to the points where its successive branches cross the axis, they will be normals to the curve.

Let (fig. 166, 167, and 180.) BE, BE′, BE″, &c. be drawn from B to the successive points E, E′, E″, &c. where the Quadratrix cuts the axis; these straight lines will be perpendicular to the curve or its tangent.

For assume (fig. 180.) the proximate radiant O*e*. It is obvious that OE is to OS as the minute arc E*s* to S*s*; but, by the last proposition, the elapsed arcs AS and A*s* are equal to the perpendiculars from E and *e*; and consequently their minute difference S*s* is equal to *te* the increment of the perpendicular radiant. Wherefore OE is to OS or OB as E*s* to *te*, and the elementary triangle E*te* is similar to the right angled triangle EOB. Hence BE is perpendicular to E*e*, and thus a normal to the curve.

Cor. Since the tangent OT is perpendicular to BE, it follows that OB or OA is to OE or the quadrant AS to OT; but OA is to OE as AS to IE, and consequently the subtangent OT is equal to the quadrant of the generating circle.

The corresponding subtangents of the second and third &c. branches would be equal to five and nine, &c. quadrants, their intervals being always equal to the whole circumference of the generating circle.

PROP. IV. PROB.

To divide a given angle into any number of equal parts by help of the Quadratrix.

Let it be required, by the description of this curve, to divide the angle AOV, (fig. 180.) suppose into three equal angles.

From the point V, where the line OV cuts the Quadratrix, let fall the perpendicular VZ, which trisect in X and Y, draw the parallels Yy and Xx to meet the curve, and join the radiants Oy and Ox; the angle AOV will be thus divided into three equal angles AOy, yOx and xOV.

For, by the genesis of the curve, while the parallels Yy and Xx advance through equal spaces ZY and YX, the radiants Oy and Ox describe equal angles.

If the perpendicular ZV were extended to the remoter branches of the curve, the trisection of it would furnish parallels which not only mark AOy, but this angle augmented successively by the third part of four right angles.

If the given angle should exceed four right angles, its radiant would evidently meet the higher branches of the curve, and the subdivision of the perpendiculars would produce a series of parallels indicating all the varieties of angular section.

The *Quadratrix* was proposed by Dinostrates for the *Subdivision of Angles*, and the *Squaring of a Circle*. Its construction, however, is difficult, and merely mechanical.

IV. LOGARITHMIC CURVE.

If perpendiculars in continued proportion be disposed at equal intervals along a given extended line, their summits will range in what is called the LOGISTIC *or* LOGARITHMIC CURVE.

Let (fig. 168.) the intervening spaces AB, BC, CD, DE, EF, &c. be all equal, and from the several points of section let perpendiculars be erected in continued proportion, or such that AL : BM : : BM : CN : : CN : DO : : DO : EP : : EP : FQ, &c.; then their summits L, M, N, O, P, Q, &c. will mark a *Logarithmic Curve.*

It is evident any intermediate points of the curve may be easily found; for bisect the space EF in G, and erect a perpendicular GH equal to a mean proportional between EP and FQ, and the extremity H will be in the curve. But the process of bisection could be repeated, and hence the curve traced out.

DEFINITIONS.

1. The extended line AF is named the *Base* of the curve.
2. Any perpendicular is an *Ordinate*, to which the distance from some given point in the base forms the corresponding *Absciss.*
3. The logarithmic curve is considered as either *ascending* or *descending*, according as it is carried upwards or downwards from the origin of the abscissæ.

PROP. I. THEOR.

The distance between two ordinates represents the Logarithm of their ratio.

The point A (fig. 168.) being the origin of the abscissæ, let DO and FQ be any two ordinates; then will DF express the ratio of B to C.

For find AB, a common measure to the segments AD and AF, divide these into portions, equal to it, and from the intermediate points B, C, E erect perpendiculars. The ordinates thus interpolated must evidently be in continued proportion; but A marking the beginning of the scale, AL will denote unit, and consequently AD expressing the number of ratios inserted between AL and DO, the interval BC will represent the number of equal ratios which connect DO with FQ; that is, the *logarithm of their ratio.*

Cor. If equal spaces *Ab*, *Ac*, *Ad*, &c. be taken on the other side of the point A, they will still denote the logarithms of the ratio of the unit AL to the ordinates *bm* and *cd*, *on*, &c.; but their opposite position intimates a decreasing series.

PROP. II. THEOR.

The subtangents at different points of the Logarithmic Curve are all equal.

Let LT and MR (fig. 169.) be drawn to touch the curve at the summits L and M of the ordinates AL and BM; the subtangent AT will be equal to BR.

For assume another pair of ordinates *al* and *bm* at equal intervals beyond AL and BM, and draw the secants *l*LT and *m*MR. From the nature of the curve, AL : *al* : : BM : *bm*; but, from the property of parallel and diverging lines, TA : T*a* : : AL : *al*, and RB : R*b* : : BM : *bm*; whence TA : T*a* : : BM : *bm*, and by division TA : A*a* : : RB : B*b*. Now, by construction, A*a* was made equal to B*b*, and therefore TA is equal to RB. This equality must obtain whatever be the proximity of the ordinates *al* and *bm* to AL and BM. At the moment, therefore, when they coalesce and the secants *l*LT and *m*MR pass into tangents, the subtangents AT and BR must be equal.

Cor. The length of the subtangent will hence determine the scale on which the curve is constructed. It may represent different systems of logarithms, according to the mutual distances of its ordinates. When the primary ordinate, or that which represents unit, is equal to the subtangent AT, the curve will exhibit the *Natural Logarithms.* For any other system of logarithms, the subtangent will be proportional to the *Modulus.*

PROP. III. THEOR.

The space included by any two ordinates of the Logarithmic Curve is equivalent to the rectangle under their difference and the common subtangent.

Let AL and CN (fig. 170.) be two ordinates, and draw LO parallel to the base; then the space ALNC is equivalent to the rectangle under ON and AT the subtangent.

For assume any intermediate ordinate BM, and another *bm* proximate to this, draw the secant *m*MR and the parallel M*r*. It is evident that RB : BM : : M*r* : *mr*, and therefore RB.*mr* = BM.M*r*; but this rectangle BM.M*r* melts into the elementary space BM*mb*, while the ordinate *bm* migrates into BM and the secant passes into a tangent; consequently the rectangles under the common subtangent AT, and the successive excesses *mr*, forming together the rectangle under AT and ON, are equivalent to the intercepted space ALNC.

Cor. Hence the whole of the descending space extended indefinitely below an ordinate, is equivalent to the rectangle under that ordinate and the subtangent. For the space included between AL and any lower ordinate is equivalent to the rectangle under their difference and the subtangent AT. When the inferior ordinate vanishes, the unlimited space will hence be equivalent to the rectangle under AL itself and AT.

PROP. IV. PROB.

To find, by the application of the Logarithmic Curve, any number of mean proportionals to two given straight lines.

Let it be required, by help of this curve, to interpolate *n* proportionals (fig. 171.) between X and Y.

Extend the ordinate AK, and make AP equal to X and AQ to Y, draw the parallels PL and QO to meet the curve, and let fall the perpendiculars LB and OE, divide the interval BE into *n*—1 parts at the points C and D, and the ordinates CM and DN will be the mean proportionals required.

For the figures AL and AO being parallelograms, the ordinates BL and OE are evidently equal to X and Y, and, from the nature of the curve, BL : CM : : CM : DN : : DN : EO.

Cor. Let AQ be double of AP, and trisect the interval BE; the ordinate CM will evidently be the side of a cube which has double the capacity of one that has AP or BL for its side. In this way, the *duplication of the cube* could in practice have easily been effected.

PROP. V. THEOR.

The radius of a circle osculating at any point of the Logarithmic Curve, is a fourth proportional to the corresponding ordinate and tangent.

Let (fig. 171.) DN and NT be an ordinate and tangent; then will the square of DN be to the square of NT as NT itself is to the radius of curvature at the point N.

For assume the equidistant proximate ordinates GH and EO, extend the secant ONT to the axis, and erect FQ perpendicular to it, meeting the ordinate DN produced in Q, draw QR, and likewise the minute lines NK and IL parallel to the axis. Since BN : GH : : LO, by conversion KH : DN : : LO—KH or 2IH : KH, and, consequently, IN : NT : : 2IH : KH. But the elementary triangle NKI or NKH being evidently similar to ITQ, NT : NQ : : KI or KH : IN, and by compounding the analogies, IN : NQ : : 2IH : IN. Now, if a circle were described through the three points N, H, and O, and the ordinate GHI were produced to meet it, the square semichord IN would be equivalent to the rectangle under the minute segment HI and the rest of the intercepted chord; whence, if these points were brought to coalesce with N, the chord of the equicurve circle, intercepted by the extension of the ordinate DN, would be equal to 2NQ. Again, from similar triangles, that chord is to the diameter of the circle as NQ to NR; whence the diameter is equal to 2NR, and the radius of curvature equal to NR. But, from the construction, DN, NT, NQ, and NR are continued proportionals, of which NR is the fourth term, or $DN^2 : NT^2 :: NT : NR$.

Cor. Hence the Logarithmic Curve has its greatest incurvation at the summit of an ordinate VY (fig. 168.) whose square is double the square of the subtangent, or the radius of the osculating circle at V is a *minimum.*

Let AB (fig. 205.) represent the ordinate VY, and the perpendicular BC the constant subtangent, join AC, draw CD at right angles to it, and erect the perpendicular DE to meet AC produced in E, the extended line AE will then exhibit the corresponding radius of curvature, and it is only requisite to prove this line to be shorter than any other obtained by following the same construction; or it may be sufficient to show that AE occupies a stationary limit, and that the proximate line ae suffers a decrement at the end a, just equal to the increment which it receives at the other end e. From the centre C describe the elementary arcs $a\alpha$, $E\varepsilon$, and Dg, extend $D\beta$ equal to dg, and draw βb parallel to CD. By similar triangles, $AB : BC :: BC : BD$, and $aB : BC :: BC : Bd$, consequently $AB : aB :: Bd : BD$, and by division, $AB : Bd$ or $BD :: Aa : Dd$; but $AB : BC :: BC : BD$, and therefore $AB : BD :: AB^2 : BC^2$; whence $AB = 2BD$, and in the preceding analogy, $Aa = 2Dd$. Again, the triangles

dgD and $a\alpha A$ being similar, $dD : Dg :: Aa : A\alpha$, and therefore Dg is the half of $A\alpha$; but $dD : dg :: Aa : a\alpha$, and consequently dg is the half of $a\alpha$. But, from similar sectors, $AC : CE :: a\alpha : E\varepsilon$; whence $E\varepsilon$ is the half of $a\alpha$ and therefore equal to dg. Hence the right-angled triangles $E\varepsilon f$ and $dg\delta$ are equal, and consequently $f\varepsilon$ is equal to δg or $D\beta$. Wherefore, the compound line $\delta\beta = c\varepsilon = Dg + \delta g + D\beta = 2Dg = Aa$, and thus, in the slight mutation of AE into ae, the segment $A\alpha$ is taken away from the one end, while the equal segment $e\varepsilon$ is annexed at the other end. The line AE, which represents the radius of curvature, is hence placed in the limit where it has, on the whole, neither increase nor diminution, and has therefore contracted into its *minimum*.'

The Logarithmic Curve was first proposed by Gregory of St Vincent soon after the noble discovery of Logarithms; but its leading properties were investigated by Huygens and others during the latter half of the seventeenth century. It is of extreme importance in various applications of Physical Science, and particularly in exhibiting the relations of elastic fluids.

V. CYCLOID.

If, in the same plane, a circle be conceived to roll along a straight line, any point in its circumference will describe the curve which is called a TROCHOID *or* CYCLOID.

Let a circle touching an extended straight line at A, (fig. 172.) roll onwards, till the point of contact comes again to touch at B; that point will have traced *a Trochoid* or *Cycloid.* But continuing the process of progressive rotation, the same point must evidently describe a repeated succession of equal and similar curves.

DEFINITIONS.

1. The circle of revolution is called the *Generating Circle.*
2. The portion AB of the extended line on which a whole circumference of the circle is measured out, is termed the *Base* of the Cycloid, and the middle perpendicular CD its *Axis.*
3. Any straight line HE drawn from the axis parallel to the base is called an *Ordinate* to the curve, the intercepted part DH of the axis being the *Absciss.*

If a thread or flexible line be unfolded from the convex side of any curve, its extremity will describe an INVOLUTE, *to which the principal curve is the* EVOLUTE.

PROP. I. THEOR.

The absciss of the Cycloid is equal to the versed sine of the corresponding arc of the generating circle, and the ordinate is equal to the sum of that arc and of its sine.

While the generating circle rolls from A to F (fig. 172.), its tracing point will come into the position E, and the segment AF of the base will evidently be equal to the elapsed arc EF. But when the circle has reached the middle position C, the remaining part FC will be equal to the supplemental arc EI or GD, and the figure EFCG is obviously a parallelogram. Wherefore the absciss DH is the versed sine of the arc DG, and the ordinate EH is equal to EG and GH, that is, to the arc DG together with its sine GH.

PROP. II. THEOR.

The supplemental chord of the generating circle is a tangent to the Cycloid.

Let FEI (fig. 172.) be the position of the generating circle corresponding to any point E of the Cycloid; the supplemental chord EI will touch the curve.

For assume in the Cycloid the point *e* proximate to E, and draw *egh* parallel to EGH.

The minute arc G*g* may be considered as coincident with a tangent to the circle at G, and consequently the angle *g*G*k* which it makes with the inflected line GD is equal to the angle GCD in the alternate segment; but this angle GCD is evidently equal to DGH, which again is equal to G*kg*. Wherefore the elementary triangle G*gk* is isosceles, and the side G*g* equal to *gk*. But, by the last proposition, the segments EG and *eg* of the cycloidal ordinates are equal to the circular arcs DG and D*g*, and consequently EG is equal to *eg* and G*g*, or *eg* and *gk*, that is, equal to *ek*. Wherefore the secant E*e*, in merging into a tangent, becomes parallel to DG, and thus coalesces with the supplemental chord EI.

The same result is obtained from another consideration. Viewing the generating circle as a polygon of innumerable sides, the Cycloid would consist of a series of minute circular arcs described from the successive points of the base on which the corners of the polygon turn. The point F is hence the centre of the arc E*e*, whose tangent EI being perpendicular to the radius FE, must therefore be contained within the semicircle.

Cor. Hence to apply a tangent to the Cycloid that shall be parallel to a given straight line. From the vertex of the curve, draw the chord DG, in the middle circle, parallel to the given line, and extend the ordinate HGE; a tangent at E must evidently be parallel to DG, and will therefore answer the condition required.

PROP. III. THEOR.

An arc of the Cycloid measured from the vertex is double of the corresponding chord of the generating circle.

The cycloidal arc DE (fig. 172.) is equal to twice the chord DG of the circle described on the axis.

For let *egkh* be the proximate ordinate to EGH, and from D as a centre describe the portion of a circle *go*.

It is manifest that *go* being considered as a perpendicular to DG, must bisect the base G*k* of the isosceles triangle G*gk*. Wherefore the elementary arc E*e*, which is equal to the segment G*k* of the parallel chord, is always the double of G*o*, or of the excess of DG above D*g*; and, consequently, estimating from the vertex D, the aggregate of the minute portions E*e* constituting the arc DE must be double of all successive increments G*o* which compose the chord DG.

Cor. Hence the length of the arc AED of the Cycloid is equal to double the diameter CD of the generating circle, and the whole cycloidal arc ADB is four times that diameter.

Scholium. The leading properties of the Cycloid might be derived with great facility from the consideration of motion. Instead of the generating circle, substitute a polygon of innumerable sides, and the Cycloid will be composed, as we have seen, of a series of minute arcs described from the successive points of contact, with the elapsed chords as radii. The direction of the curve or its tangent at E must therefore be at right angles to EF, and will hence coincide with the supplemental chord EI.

But the same conclusion is attained by a different process of reasoning. For every portion of the Cycloid being described by a rotatory, combined with an equal progressive, motion, the direction of the curve at any point E must evidently be the diagonal resulting from this compound traction. But the rotatory motion is perpendicular to EO, and the progressive motion perpendicular to OF; and consequently the tangent to the curve must run perpendicular to the diagonal EF, or take the position of the supplemental chord EI.

Hence the inflexion of the Cycloid is half the inflexion of the generating circle. For when this circle has arrived at the contact F, the describing radius has moved from the position OF into OE, and consequently the circumference at E has bent through an angle FOE; while the curve of the Cycloid has only turned from the direction FI into EI, making an inflexion OIE which is the half of the exterior angle FOE. The Cycloid thus rises perpendicularly at A, becomes parallel to the base at the vertex D, and again descends perpendicularly at B.

An arc of the Cycloid, reckoning from its vertex, being thus double of the corresponding chord of the generating circle, which is itself equal to twice the sine of half the angle of rotation; those cycloidal arcs must be proportional to the sines of inflexion at every point of the curve.

It has been proved that OF is to EF as the velocity of rotation is to the velocity with which the Cycloid is described at E, or as each increment of the circumference of the generating circle is to the corresponding elementary arc of the cycloid. Draw E*m* perpendicular to the radius OE, and OF : EF : : E*e* : E*m*; but the elementary triangle E*em* being similar to the isosceles triangle EOF, the parallel *en* to EF must bisect the base E*e*, and therefore this element of the cycloidal curve DE is double of the corresponding increment E*n* of the supplemental chord EI. Now the same relation subsisting at every step from D to E, and from D to G, the aggregate arc DE must be double of the accrescent chord DG or IE.

PROP. IV. THEOR.

The exterior space between the Cycloid and its vertical tangent is equivalent to the segment of the generating circle intercepted by the ordinate.

Let LDM (fig. 172.) be a tangent applied at the vertex of the cycloid, and EHN a double ordinate drawn; complete the rectangle ELMN, and the area of the exterior curvilineal space EDNML will be equal to that of the corresponding circular segment GDK.

For assume the proximate ordinate *he*, and extend it to LE, and parallel to it draw *le*.

Because the minute cycloidal arc E*e* may be considered as a portion of the tangent at E, which is parallel to the chord DG; the elementary triangle E*se* is evidently similar to the right-angled triangle DHG. Wherefore DH or LE : GH : : E*s* or H*h* : *se*, and LE.*se*=GH.H*h*, or the exterior elementary rectangle L*e* is equivalent to the rectangle G*h* within the circle. Consequently the exterior space ELD, formed by those accumulated rectangles L*e*, is equivalent to the circular segment GDH, which is composed of the aggregate internal rectangles G*h*. The double external space EDNML must therefore be equivalent to the double of GDH, or to the complete segment GDK.

Cor. Hence the whole area of the Cycloid is triple of that of the generating circle. For the circle being equivalent to the exterior space, the area of the Cycloid must be the excess of the rectangle under its base and axis above the area of that circle. But the base and axis being the same as the circumference and diameter of the generating circle, the rectangle is four times the area of this circle, and consequently the whole cycloidal space is three times that area.

PROP. V. THEOR.

If a thread or flexible line, attached to the end of a Cycloid and bent along the convexity of the curve to its vertex, be then unfolded, its extremity will describe another half cycloid, equal and similar to the former.

Let APK (fig. 173.) be a semi-cycloid, of which LA is the axis and LK half the base; and conceive a line, fastened at K, and applied externally to the curve as far as the vertex A, to be again unfolded, its remote end will trace from A to D a similar and equal curve: and after it has reached D, and again bends along an opposite corresponding semi-cycloid KB, it will describe the complete Cycloid ADB.

For draw ACB parallel to the compound base LKM of the two semi-cycloids. When the thread is completely unfolded, it will touch both curves at K, and hence occupy the perpendicular KCD, which being thus equal to the arc APK, is double of the diameter AL of the generating circle, and consequently CD is equal to CK or AL. On AL and CD describe equal semicircles, let KPE be an intermediate position of the folding line, draw PON and EGH parallel to AC, and join AO, OI, CG and DG.

Since PE touches the curve at P, it is parallel to the chord AO of the generating circle, and consequently AOPQ is a parallelogram, and the opposite sides AO and PQ, AQ and OP are equal. Now the line PE, being equal to the cycloidal arc AP, is double of the chord AO;

whence EQ is equal to PQ or AO. Wherefore, from the property of parallel and diverging lines, CH=CR=AN; consequently, if the semicircle DGC were transferred to LOA, the points D, H and C would coincide with L, N and A, and the perpendicular HG with NO; wherefore the chord CG is equal to AO or QE, but it is likewise parallel to QE, since the angle HCG is equal to NAO or AQE; whence EGCQEG is a parallelogram, and the opposite side EG is equal to QC. But AQ being equal to the circular arc AO, and AC to the semi-circumference AOL, the segment QC must be equal to the remaining arc OL, and consequently EG is equal to the corresponding arc DG. Wherefore E is a point of a Cycloid having its generating circle described on the diameter CD.

From the uniform genesis of this curve, it follows, that all Cycloids are cast on the same mould, and are therefore similar figures.

The Cycloid will assume endless varieties of form, if the tracing point attached to the generating circle be taken without or within the circumference, though still in the same plane. In the former case, the curve having its base shortened, is called the *Curtate* or *Contracted Cycloid*, and has a nodated convolution, (fig. 174.) But when the tracing point lies within the generating circle, the curve having its revolution distended is named the *Prolate* or *Inflected Cycloid*, and acquires an undulating shape, (fig. 175.)

The properties of these modified cycloids are nearly related to those of the principal curve, from which they might easily be derived. But, as it would not suit the nature of an elementary work to engage in such details, I shall confine this analysis to a curve closely related to the Cycloid, and therefore called the *Companion to the Cycloid*.

VI. COMPANION TO THE CYCLOID.

If the ordinates of the Cycloid be diminished by the corresponding ordinates of the generating circle, the curve will be transformed into the COMPANION *to the* CYCLOID.

Let (fig. 176.) BAC be a Cycloid, of which the generating circle is described from the centre O, and HE an ordinate; if HD be taken equal to FE, the point D will mark out the *Companion to the Cycloid.*

It is hence obvious, that the ordinate ED of this curve must be equal to AF, the corresponding arc of the generating circle.

This curve will evidently form a series of equal convolutions about an extended axis drawn through O parallel to BC.

PROP. I. PROB.

To apply a tangent at any point of the *Companion of the Cycloid.*

Let it be required to draw (fig. 176.) a straight line touching the curve called the *Companion of the Cycloid* at the point D.

Assume any proximate point *d*, draw the secant D*d*T the ordinates DFE and *dfe*, and the parallels *d*∂ and *f*φ, and join the point F with the centre O of the generating circle.

From the nature of this curve, the ordinates DE and *de* are equal to the circular arcs AF and A*f*, and consequently the minute difference Dδ is equal to the elementary arc F*f*. Wherefore E*f*φ being considered as an elementary triangle similar to FOE, it follows, that F*f* or Dδ : *f*φ or *d*δ : : FO : FE ; but the elementary triangle D*d*δ being obviously similar to DTE, Dδ : *d*δ : : DE : TE, and consequently FO : FE : : DE : TE. Make EG equal to the radius FO, and EG : FE : : DE : TE; join FG, and the right-angled triangles FEG and DET are similar; whence the angle EDT is equal to EGF, and therefore DT, in merging into a tangent, is perpendicular to the line FG.

Cor. Hence, proceeding from the vertex where the curve runs parallel to its base, the curve gradually bends nearer the direction of the axis till it reaches the middle position, and then reclines again towards its first direction. For EG continuing equal to this radius, the angle EGF, or the inclination of the tangent to the base of the curve, must increase till the ordinate DE passes through the centre, when that angle becomes half a right angle. Beyond this position the angle EGF will diminish till the curve grazes the base. At its middle position the curve must therefore suffer a contrary flexure.

PROP. II. THEOR.

The area of a segment of the *Companion to the Cycloid*, is equivalent to the excess of the rectangle under the absciss and the sum of the corresponding arc and sine of the generating circle, above twice the area of the circular segment.

The curvilineal segment DAE (fig. 176.) is equivalent to the rectangle under AE and the sum of the arc AF and its sine EF, diminished by the circular segment FAE.

For let the cycloid BHAC stand on the same base, and produce the ordinate ED and the proximate ordinate *ed* to meet that curve in H and *h*. Since HF is equal to the circular arc AF, it is equal to DE; whence the elementary rectangle H*hf*F is equal to the rectangle D*de*E, and therefore the whole exterior space HAF is equivalent to the segment DAE of the *Companion to the Cycloid.* But the space HAF is only the rectangle HIAE, or that under AE and HE, diminished on the one side by the circular segment DAE and on the other by the equivalent space HIA. Wherefore, the curvilineal segment DAE is equivalent to the rectangle under AE and the sum of AF and EF, diminished by twice the circular segment DAE.

Cor. Hence the whole area of the Companion of the Cycloid is double that of the generating circle. For the rectangle BIAL being four times the area of the semicircle AFL, leaves BDAL equivalent to double that space; but the same relation subsists on the opposite side of the diameter AL, and consequently the whole surface BDAC of the curve is double the area of the generating circle.

PROP. III. PROB.

To find an aliquot part of an angle by the application of this curve.

Let it be required to find (fig. 176.) any part, suppose the third, of the angle AON.

About O describe a generating circle and construct the *Companion of the Cycloid.* Draw the ordinate KNM, from which cut off the third part KP, draw PQ parallel to the diameter and terminating in the curve, then draw

RPQ parallel to the base and cutting the circle in R, join OP, and the angle AOP is the third part of AON.

For the arc APN being equal to the ordinate KNM, the arc AP is equal to QO or KN, the third part of that ordinate.

Scholium. Pursuing the analogy, it is obvious, that the *Companion to the Cycloid* must include a succession of waving branches, extending on the same side of the diameter of the generating circle. The ordinates corresponding, for instance, to the absciss AK, are AM,Am,AM′,Am', AM″, Am'', &c. equal to the arcs AR, ARLS, and these continually augmented by a whole circumference. It hence follows, that the section of the same angle involves as many different results as the number of the divisions.

The Cycloid has afforded the finest scope for the exercise of the Modern Geometry. Its properties are beautiful, prolific, and highly important in the Theory of Dynamics. This curve seems to obtrude itself on our observation; and Aristotle has incidentally noticed it, in alluding to the paradox, that a nail on the rim of a wheel describes a longer line than the axle. The Cardinal Cusano proposed the Cycloid for the quadrature of the circle, and Galileo contented himself with mentioning it in vague and general terms. Roberval was the first who assigned the simpler properties of this curve. In 1634, he discovered its area; but Fermat and Des Cartes immediately solved that problem, and likewise applied tangents to the curve. Roberval resumed the investigation, and struck out his expeditious method of drawing tangents generally, by tracing the diagonal of a compound motion. Torricelli, the very ingenious inventor of the Barometer, apparently unconscious of what had been already done on the subject, wrote, in 1644, shortly before his premature death, a neat Geometrical Tract, embracing the chief properties of the Cycloid.

Other mathematicians now entered the career of discovery. Wallis made considerable advances; but the great Pascal, by a challenge in 1658, drew the attention of the scientific world to the higher properties of the Cycloid. This was partly answered by Slusius, Wren, Wallis, and Lalouere.

The name of Huygens has derived unfading lustre, chiefly from his fine discoveries on the Cycloid. He rectified or found the length of this curve, as early as 1657; but pursuing his researches, he afterwards reared the beautiful theory of the *Evolution of Curves*, of which the Cycloid affords so remarkable an exemplification. He likewise detected the isochronism of the curve, or the singular property that bodies, gliding along it, will descend from any point in the same time. In the year 1673, which forms an epoch in the history of mechanics, Huygens combined those discoveries, and applied them to the improvement of the pendulum, in his *Horologium Oscillatorium*; which, as a model of clearness, elegance, and geometrical taste, still deserves the attentive perusal of the inquisitive student.

VII. EPICYCLOID.

If, in the same plane, a circle be conceived to roll upon the circumference of another circle, either externally or internally, any point of the revolving circumference will describe the curve called the EPITROCHOID *or* EPICYCLOID.

LET a circle (fig. 181.) whose centre is E touch another circle at A, and revolve on the outside of the circumference, the point of contact now carried round till it return again at F will trace an Epicycloid; or, if the circle rolls along the inside of the circumference, it will trace a curve of the same kind. When the radius EB or E′B is an aliquot part of the radius OB, the Epicycloid, after repeated accessions to the circumference of the larger circle, must evidently recommence the series at the same point A.

DEFINITIONS.

1. The circle of revolution is called the *Generating Circle.*
2. The circle on whose circumference the revolution is performed, is termed the *Fundamental Circle.*
3. The portion AF of the circumference of the fundamental circle on which the Epicycloid rests, is called its *Base.*
4. Any segment AB of the base is an *Absciss,* to which the chord BC or BC′ joining its extremity, or the point of incident contact, with the tracing point, is an *Ordinate.*

PROP. I. PROB.

Given the generating and fundamental circles of the Epicycloid, to find the ordinate corresponding to a given absciss.

Let AB (fig. 181.) a portion of the circumference of the fundamental circle whose centre is O, be an absciss of the Epicycloid, and it be required to find the corresponding point C or C′ of the curve.

Join OA, and BC or BC′, draw OE or OE′ through B the point of incident contact, make BE or BE′ equal to the radius of the generating circle, and describe it from E or E′, and lastly join EC or E′C′.

From the nature of the curve, the arc AB of the fundamental circle must be equal to the arcs BC or BC′; and therefore the radius of the generating circle, BE or BE′, is to BO, as the angle AOB to BEC or BEC′. Hence make the angle BEC or BEC′ to AOB as the radius of the generating to that of the fundamental circle, and C or C′ will be the corresponding points of the Exterior or Interior Epicycloid.

This construction is always effected geometrically, if the radius EB be an aliquot part of OB. When the arc AB becomes equal to the semicircumference of the generating circle, the points C or C′ will evidently coincide with D or D′, the vertex of the curve, and occupy the remotest or the nearest position in respect to the centre of the fundamental circle.

PROP. II. THEOR.

A line drawn through any point of an Epicycloid perpendicular to the corresponding ordinate, is a tangent to the curve.

If a perpendicular be erected at C or C′ (fig. 181 and 184.) to the chord CB or C′B joining the corresponding contact of the two circles, it will touch the Epicycloid.

For, instead of those circles, conceive two polygons consisting of numerous sides all equal to B*b*. The generating polygon would then describe a series of minute circular arcs having the successive points of contact for their centres. The extremity C of the chord BC would first turn on the point B and next on *b*. The tangent to the minute arcs thus traced, and which finally melt into the Epicycloid, is hence the perpendicular to their radius BC or BC′.

Cor. The tangent to the Epicycloid may be considered as an extension of the supplemental chord. For the angle BCD or BC′D′ being contained in a semicircle is a right angle, and consequently the supplemental chord DC or D′C′, which is perpendicular to BC or BC′, must touch the curve.

PROP. III. THEOR.

The radius of the fundamental circle is to twice the sum of the radii of both circles in the Exterior Epicycloid, but to their difference in the Interior Epicycloid, as the supplemental chord in the generating circle is to the length of the corresponding portion of the curve from its vertex.

Let FG (fig. 204.) be the axis of the curve, and B the contact of the generating circle in its oblique position, from the centre O describe a circle through C to I, draw the chords BC and FI and the supplemental chords DC and GI; then OB is to twice the sum of OB and BE or 2OE, as GI or DC to the arc GC of the Exterior Epicycloid, and as OB to 2OE′ as D′C′ to the arc KC′ of the Interior Epicycloid.

For conceiving the circles as before to be only polygons composed of innumerable sides B*b*, the tracing chord BC of the exterior generating circle, in turning about B till the proximate point *b* falls on the mutual tangent, will first describe an angle equal to BE*b*; and will next describe an angle equal to BO*b*, in bringing that point to the circumference of the fundamental circle. In the Exterior Epicycloid, therefore, the chord BC describes at every stage an angle CB*c*, equal to the sum of the elementary angles BE*b* and BO*b*.

But in tracing the Interior Epicycloid, the chord BC′ does not carry the proximate point so far as the mutual tangent, but stops at the circumference of the fundamental circle. The angle C′B*c*′ is in this case, therefore, only the difference of the elementary angles BO*b* and BE′*b*. Now, very small angles being as their sines, the angles BO*b* and E*b*O and E′*b*O are evidently as OB to OE or OE′, their opposite sides. But these angles being formed at the centres O, and E or E′, of the circles, equal angles at the circumference would stand on double arcs. Make C*d*=B*b*, and having joined the chord Cδ*d* and the supplemental chord D*d*, it follows, that OB : 2OE or 2OE′ : : Cδ : C*c* or C′*c*′, and therefore OB is to 2OE or 2OE′, as the aggregate of the increments Cδ, which constitute the supplemental chord CD, is to the cumulative amount of the elementary arcs C*c* or C′*c*′ which compose the arc CG or C′K of the Epicycloid.

Cor. 1. Hence OB is to 2OE or 2OE′, as twice the versed sine of half the corresponding arc BC in the generating circle is to the portion AC of the Epicycloid. For OB : 2OE or 2OE′ : : CD : CG, and likewise OB : 2OE or 2OE′ : : BD : ACG, whence OB : 2OE or 2OE′ : : BD—CD : AC; but the difference between the supplemental chord and the diameter BD is evidently twice the versed sine of half the arc BC.

Cor. 2. If the diameter of the generating circle be an aliquot part of the diameter of the fundamental circle, the whole epicycloidal circuits are capable of being expressed geometrically in terms of those lines. For let BE be denoted by 1, and BO by n; then $n : 2n \pm 2 : : 2 : AG$, or $n : 2n \pm 2 : : 4 : AGH$. Whence the length of each complete epicycloidal arc is denoted by $8\left(\frac{n \pm 1}{n}\right)$, and the extent of the series of those arcs clustering round the fundamental circle is equal to $8(n \pm 1.)$

Cor. 2. Let $n = 1$, and $8\left(\frac{n+1}{1}\right) = 16$; or the length of the Exterior Epicycloid described by a circle about an equal circle is 8 times their diameter. This is the caustic curve produced by reflexion from a circle when the radiant point lies in the opposite circumference. (*See fig.* 183.) The Interior Epicycloid in this case evidently degenerates into a point, and accordingly $8(n-1) = 0$.

Let $n = 2$, and the length of the Exterior Epicycloid is $8\left(\frac{n+1}{n}\right) = 12$, or 6 times the diameter of its generating circle. This is the caustic curve of parallel rays in the circle. (*See fig.* 182, where the half Epicycloid only is represented.) The length of the Interior Epicycloid is in this case $8\left(\frac{n-1}{n}\right) = 4$, or the diameter of the fundamental circle which is then actually traced.

Let $n=3$, and the extent of the Exterior Epicycloid is $10\frac{2}{3}$, of its three clustered branches amounts to 32; the corresponding lengths of the Interior Epicycloid, and of its repeated bends are $5\frac{1}{3}$ and 16, or the halves of the former.

Let $n=4$, and the lengths of the Exterior and Interior Epicycloids will be respectively 5 and 3 times the diameter of the generating circle, as in fig. 181.

When n becomes indefinitely great, $8\left(\frac{n\pm1}{n}\right)=8$, or the length of this Epicycloid, whether Exterior or Interior, is equal to four times the diameter of the generating circle; the curve now passes into the common cycloid, the fundamental circle expanding into a straight line.

PROP. IV. THEOR.

As the radius of the fundamental circle is to that radius joined to twice the sum or twice the difference of the radii of both circles, so is any segment of the generating circle to the corresponding segment of the Exterior or Interior Epicycloid.

Let the generating circle roll from A to B, (fig. 204.) then as OB is to 2OE+OB or 3OB+2BE, so is the circular segment CMB to the segment ABC of the Exterior Epicycloid; but as OB to 2AE+OB or 3AB—2BE, so is the circular segment C′NB to the segment ABC of the Interior Epicycloid.

For, assuming the proximate point b, it was shown that the arc Bb is to Cc, or the angle BCb to CBc, as OB to 2OE or 2OE′; wherefore the elementary sector BCb is to

CBc in the same ratio, and consequently OB is to the sum of OB and 2OE or 2OE′, as BCb′ to the compound accrescence BCcb or BC′c′b′. But the accumulative sectors BCb form the circular segment BMC, and the elementary spaces BCcb, BC′c′b compose, by their aggregation, the segment ABC or ABC′ of the exterior or interior epicycloid; whence OB : 3OB+2BE : : BMC : ABC, and OB : 3OB—2BE : : BNC : ABC′.

Cor. When the radius of the generating circle is an aliquot part of the radius of the fundamental circle, the entire epicycloidal space may be expressed numerically in terms of the areas of those circles. Let BE$=1$ and OB $=n$; the surface of the Exterior and that of the Interior Epicycloid will be $\frac{3n+2}{n}$ times, and $\frac{3n-2}{n}$ times, the area of the generating circle, and consequently the agglomerate epicycloidal spaces clustered about the fundamental circle will be $3n+2$ and $3n-2$ times that same area. Suppose $n=1$, and the Exterior Epicycloid must contain five times as much space as the generating circle. (*See* fig. 183.) Let $n=2$, and the Exterior Epicycloid will be quadruple, and the interior only double of the generating circle. If $n=3$, the cluster of Exterior and Interior Epicycloids will respectively amount to 11 and 5 times the area of the generating circle. When $n=4$, the clustering of epicycloidal spaces, exterior and interior, will be 14 and 10 times greater than the generating circle, or each of those spaces will be 3½ and 2½ times that circle. (*See* fig. 181.)

If n be considered as unbounded, both the expressions $\frac{3n+2}{n}$ and $\frac{3n-2}{n}$ will become equal to 3; so that, when the Epicycloid passes into a common Cycloid, it contains only triple the area of the generating circle.

PROP. V. THEOR.

The sum or difference of the radius of the fundamental circle and twice the radius of the generating circle, is to the sum or difference of twice the radius of the fundamental circle and twice the radius of the generating circle, as any chord of this circle is to the corresponding radius of curvature of the Exterior or Interior Epicycloid.

Let BC (fig. 206.) be the position of the tracing chord, and R the centre of the circle which osculates at C or C″; then OB+2BE : 2OB+2BE : : BC : RC, in the Exterior Epicycloid; and OB—2BE : 2OB—2BE : : BC′ : RC′, in the Interior Epicycloid.

For assuming the proximate point *b*, and assigning the elementary arc C*c* or C′*c*′ as before, the tracing chords B*c* and B*c* or B*c*′ must be perpendicular to the curve at the points C and *c* or *c*′, and consequently being extended backwards, they will meet in R or R′, the centre of the osculating circle. Wherefore in the triangle *d*CR, as B*d* or BC is to *d*R or CR, so is the sine of the angle BR*d* to the sine of the angle RB*d* or CB*d*; but the sines of such minute angles are as the angles themselves, and consequently BC is to CR as the angle BR*d*, or the excess of the angle CB*d* above B*d*R, is to the angle CB*d* or CB*c*. But the angle CB*c* is to B*d*R or B*db* as the minute arc C*c* is to B*b*, and therefore the angle BR*d* is to CB*c* or BC is to CR, as the excess of C*c* above B*b* is to C*c*. Now, it was shown, that B*b* : C*c* : : OB : 2OE, and consequently, $Cc-Bb : Cc :: 2OE-OB$, or $OB+2BE : 2OE$, or $2OB+2BE$, and $BC : CR :: OB+2BE : 2OB+2BE$, in the Exterior Epicycloid.

The same reasoning applies to the Interior Epicycloid, since $Bb : C'c' :: OB : 2OE'$, and hence $C'c'-Bb : C'c' :: 2OE'-OB$ or $OB-2BE : 2OE$, or $2OB-2BE$, wherefore $BC' : C'R' :: OB-2BE : 2OB-2BE$.

Cor. Suppose the radius BE or BE′ of the generating circle $=1$, the radius of the fundamental circle $=n$, and the tracing chord of the generating circle $=c$; then the radius of curvature belonging to the Exterior and Interior Epicycloids will be expressed by $c.\frac{2n+2}{n+2}$ and $c.\frac{2n-2}{n-2}$.

If $n=1$, the radii of curvature in the Exterior and Interior Epicycloids are $\frac{4}{3}c$. and $\frac{1}{2}c$. (*See* fig. 183.)

When $n=2$, the radius of curvature in the Exterior Epicycloid is $\frac{3}{2}c$; but in the Interior Epicycloid it becomes infinite, the curve now degenerating into a straight line, the redoubled diameter of the fundamental circle.

If $n=3$, the radii of curvature of both Epicycloids are $\frac{8}{5}c$, and $4c$. Such are those represented in fig. 206.

If $n=4$, the radii of curvature are respectively $\frac{5}{3}c$, and $3c$. (*See* fig. 181.)

When n becomes indefinitely large, these Epicycloids pass into the common cycloid, of which the radius of curvature is $2c$, or twice the tracing chord, as has been already shown.

PROP. VI. THEOR.

The Involute of an Exterior Epicycloid is a similar curve.

Suppose a thread fastened at A (fig. 184.), the origin of the Exterior Epicycloid ACG, and bent along the convex side of the curve as far as the vertex G, were again to unfold itself, the extremity H would describe another similar Epicycloid.

For let the generating circle arrive at B, and the tangent to the Epicycloid at the point C, or the evolving thread, will form an extension CDH of the supplemental chord DC equal to the Epicycloidal arc CG; but OB is to 2OE, or to the sum OB and OD as CD to CG or CDH, and by division OB is to OD as CD is to DH. Draw HI parallel to BC, and the triangle DHI being consequently similar to DCB is right-angled and contained in a semicircle, of which let K be the centre. But, from the property of parallel and diverging lines, CD : DH : : BD : DI; whence OB : OD : : BD : DI, and alternately OB : BD : : OD : DI, and by composition OB : OD : : OD : OI. If, therefore, a circle be described from the centre O through G and D, the circle IHD will bear the same relation to it that the circle BCD has to ABF. But the rolling of the circle DHI upon DG traces the same curve as the unfolding of the thread CDH; for the radius OB is to OD as the arc CD is to DH, and as the arc BF is to DG, consequently, since the arc CD is equal to BF, the arc DH must be equal to DG.

To obtain the involute, therefore, describe a fundamental circle from the centre O through the vertex G, and make OF : OG : : BD : DI, the diameter of the generating circle, which is to roll upon DG.

Scholium. If the centre O be conceived to be infinitely remote, the ratio of OB to OD will merge into that of equality, and the new generating circle will become equal to the former; the resulting curve will hence be similar and equal to ACG, but described on a parallel rectilineal base. Such is the involute of the common Cycloid.

The Cycloid may thus be always considered as merely the extreme limit of the Epicycloid, from which it derives by simplification all its elegant properties.

The Epicycloid likewise includes the *Caustic Curve*, produced by the reflexion of light from the circumference of a circle. When the rays issuing from any point fall upon the concavity of a circle, the several reflected pencils are collected into a series of brilliant points or *foci*, which form, by their assemblage, a bright curved track.

First, Suppose *parallel* rays to strike the interior circumference of a semicircle, and let it be required to find the path of their concourse after they have suffered reflexion. Draw the diameter AOC (fig. 182.) at right angles to the adjacent rays RE and *re*, and erect the perpendicular radius OD, join the points of incidence I and *i* with the centre O; the reflected rays IF and *i*F, making the angles EIF and E*i*F equal to OIR and O*ir*, will meet in a focus F. Wherefore the angles RIF and *ri*F will be the double of RIO and *ri*O, and since RE and *ri* are parallel, the difference IF*i* of their inclinations is hence double of the central angle EO*e*. But a circle being described through the three points I, *i* and F, and consequently touching the primary circle in I, the angle IEi at the centre E being double of the angle IFi at the circumference, must be four times greater than IOi. Whence the radius EI is the fourth part of OI, which is thus bisected in G. With OG, therefore, as a radius, describe the interior semicircle BHI, and the angle FIG being equal to the incident angle OIR, is equal to the alternate angle IOH or GOH; whence the arc GF intercepted by the angle GIF at the circumference is equal to the arc GH intercepted by the angle GOH at the centre of a circle whose radius OB is twice GE. If the small circle, therefore, had rolled on the outside of the quadrant BGI, when it gained the contact G, its tracing point would have occurred in F. Whence, the chain of foci which constitutes the caustic curve of parallel rays is the Exterior Epicycloid ALBFC traced on the semicircumference of a circle, having the same centre as the primary circle but only half the radius, by another circle still one-half smaller.

Next, Suppose the radiant point R (fig. 183.) to lie in the circumference of the primary circle. Extend the diameter ROD, join the points of incidence I and i with the centre O, and draw IF and iF making the angles OEF and OiF equal to RIF and RiF, and their concourse F will be the focus of the incident rays RI and Ri. The angle IFi must evidently be the excess of twice the angle IRi above IRi, that is, the triple of IRi; and consequently the diameter IG of the circle which contains the angle IFi, or passes through the three points I, i and F, is the third part of the diameter RD of the primary circle. Wherefore the radii EI and EG are each the third of OI or OD, and a circle described with the radius OG is equal to the circle GFI. But the angle BOG is double of ORE, and GEF is double of the equal angle OIF; whence these equal central angles BOG and GEF intercept equal arcs BG and GF. The circle GFI having the third part of the diameter of the primary circle, and rolling on the circumference of an equal circle, will describe an Exterior Epicycloid RIFBR, which is the caustic curve belonging to the radiant R.

The extension of the principle of the Cycloid naturally gave rise to the Epicycloid. This curve was immediately rendered subservient to the purposes of Practical Mechanics; the Danish Astronomer, Römer, to whom we are indebted for the fine discovery of the progressive motion of light, having proposed it, during his residence at Paris, as the proper form of the *teeth of wheels.* The rectification of the Epicycloid appeared in the first edition of the *Principia* in 1687. Its quadrature, indicated by Caswell, was afterwards demonstrated by Halley. But the higher properties of the curve were discovered before the close of the seventeenth century, by the illustrious John Bernoulli. Lahire, in 1704, composed a diffuse general treatise on Epicycloids.

The *Caustic Curves*, which may be considered as analogous or supplementary to the Epicycloids, were invented by Tschirnhaus, who likewise detected several of their more remarkable properties. But the activity and penetration of John Bernoulli soon exhausted the subject.

VIII. PARABOLOID.

If a curve assume for its absciss and ordinate the terms of a continued proportion, of which the first is a given line or parameter, it will exhibit a more general form of the Parabola, which is therefore called a PARABOLOID.

To avoid the multiplicity of details, I shall only select the most remarkable of these curves; whose ordinate is the second of two mean proportionals inserted between the parameter and the corresponding absciss, and which has commonly received hence the name of the *Semi-Cubical Parabola*.

Let AB (fig. 194.) be an absciss, and PB the parameter; if the continued proportion, PB : QB : : QB : RB : : RB : AB be pursued, and the ordinate BC made equal to RB the third term, this species of Paraboloid will be formed.

If the second term QB should change its position into Q′B, the succeeding terms will be alternately transposed; while the third term RB, which corresponds to BC or B′C′, will retain its place, the fourth term AB will be thrown to the other side, forming the absciss AB′. Hence the curve consists of two branches standing upon the axis, united in a common vertex, but extending indefinitely in opposite directions.

LEMMA.

If, while the first term of a continued proportion remains the same, the second receive a minute accession, the succeeding terms will acquire increments, as their double, triple, &c.

The terms (fig. 196.). AB, CD, EF, GH, &c. of a continued proportion would evidently arrange themselves at mutually equal distances, AC, CE, EG, &c. in a logarithmic curve. If the second term CD should be augmented to cd, a new set of terms ef, gh, &c. would arise, which occupy likewise equal intervals Ae, ce, eg, &c.; consequently the displacements Ec, Gg, &c. must be double, triple, &c. of Ec. But, from the property of the logarithmic curve, the subtangent TE is to the several ordinates CD, EF, GH, &c. as the displacements Cc, Ee, Gg, &c. to the increments κd, εe, γg, &c.; or alternately TE is to Cc, Ee, Gg, &c. as CD, EF, GH, to κd, εe, γg, &c.; wherefore these successive augments δd, εe, γg, &c. are proportional to CD, 2EF, 3GH, &c.

PROP. I. THEOR.

A tangent to the Paraboloid meets the axis at a distance beyond the vertex, equal to half of the corresponding absciss.

Let the straight line CD (fig. 194.), touching the Paraboloid at the point C, be produced to meet the axis in D; the exterior portion AD of the subtangent will be equal to half of the part AB included within the curve.

For draw the proximate ordinate *bc*, and consider CD as a secant joining the contiguous summits C and *c*. Since BC and BA are the second and third terms of a continued proportion, of which the given parameter is the first term, their increments βc and $C\beta$ are, by the preceding lemma, as 2BC and 3BA. But, from similar triangles, $\beta c : C\beta :: BC : BD$, and consequently $BC : BD :: 2BC : 3BA$; wherefore $2BD = 3BA$, and $BA = 2AD$ or $AD = \frac{1}{2}BA$. Hence, when the secant melts into a tangent, the exterior portion AD must be half of the abscis BA.

PROP. II. THEOR.

The area of the Paraboloid is three-fifth parts of its circumscribing rectangle.

Any segment BAC (fig. 194) of the curve occupies three-fifth parts of the space contained by the circumscribing rectangle AGCB.

For draw the tangent CD, extend the rectangle DECB, assume the proximate ordinate *bc*, and complete the rectangle D*ecb*.

The complements C*e* and C*b* of the parallelograms constituted upon the diagonal CD are equivalent; but the elementary rectangle C*g* is to C*e* as CG to CE, or as BA to BD, that is, as 2 to 3. Wherefore the rectangle C*g* is two-third parts of C*b*, and consequently its accumulated amount, or the exterior space AGC, is equivalent to two-thirds of the segment ACB; whence this area itself must be three-fifths of the whole circumscribing rectangle AGCB.

PROP. III. THEOR.

The Paraboloid is the evolute of the common parabola.

The centres of all the circles which osculate with the parabola range themselves in a Paraboloid; or if a thread applied to this curve, and extended to a certain distance beyond its vertex, be then unfolded, it will describe the common parabola.

Let HAE (fig. 195.) be a parabola, of which F is the focus. The radius of curvature at the vertex A being half of the parameter of the axis, if FD be made equal to AF, the point D will be the centre of osculation. To find an osculating circle to any other point C of the curve, since it cuts off a chord equal to the parameter of the corresponding diameter, make CM equal to double the focal distance FC, and erect the perpendicular MO, which will meet the extended normal CO in O the centre of that circle. It only remains to be proved that O is a point of the Paraboloid.

For draw the tangent CT of the parabola, and from T erect TL perpendicular to the axis, meeting OC produced in the point L; draw likewise OHI parallel to the axis, cutting the perpendiculars DH and CBI in H and I.

From the property of a tangent of the parabola, FT is equal to FC and to FG; whence TG is double of FC, and therefore equal to CM or IO. The right triangles LTG and CIO having the side TG equal to IO, and the oblique angle TLG equal to its alternate angle ICO, are hence equal; wherefore the hypotenuse LG is equal to the radius of curvature CO, or taking CG from both, the segment LC is equal to GO.

Again, since the subnormal BG and AD the radius of vertical curvature, are each equal to the semiparameter of the axis, take away the common part BD, and DG remains equal to AB and consequently equal to half of the subtangent TB. Wherefore, from the property of parallel and diverging lines, GK is the half of LC or of GO, and hence DK is likewise the half of DH. But this constitutes the distinguishing character of the Paraboloid, to which OK is therefore a tangent, and OH and DH an ordinate and absciss.

The ordinate and absciss of the Paraboloid are easily found. For HK being the triple of DK, it is evident that OH must be the triple of AB the absciss of the parabola. Again, since LC is equal to GO, it is evident that LN must be equal to BI or DH; but, from similar triangles, BG or 2AF : BC : : FB or 2AB : LN, or alternately AF : AB : : BC : LN or DH, the absciss of the Paraboloid.

The parameter of the Paraboloid may likewise be assigned. Since DH : AB : : BC : AF, therefore DH : 3AB or OH : : BC : 3AF, and consequently $DH^2 : OH^2$: : BC^2 or 4AF.AB : $9AF^2$: : 4AB : 9AF : : 8AB or 4TB : 18AF or 9BG : : 4LC or 4GO : 9CG : : 12GO : 27CG. But GO being two-thirds of OK, 12GO is equal to 8OK; whence $DH^2 : OH^2$: : 8OK : 27CG : : 8OH : 27BG. Make 8AV=27BG; whence 8 to 27, as BG, the semiparameter of the axis, to AV, and $DH^2 : OH^2$: : 8OH : 8AB : : OH : AV. Wherefore AV is the parameter of the curve, being the first term of a continued proportion, of which OH and DH are the third and fourth terms.

PROP. IV. PROB.

To determine the length of any portion of the Paraboloid.

Let it be required to measure an arc OD (fig. 195.) of the Paraboloid.

Since the evolution of a line from the curve would describe a parabola, it is evident that the arc OD must be equal to the excess of the radius of curvature OC above the radius DA of the circle osculating at the vertex A, or the semiparameter of the axis. It only remains to find the parabola.

From similar triangles, KH : OH : : BC : BG, and consequently $KH^2 : OH^2 :: BC^2 : BG^2 ::$ BT or 2DG : BG. Wherefore, produce the absciss HD till DK be equal to the half of it, and draw the tangent OK to the Paraboloid; make $DK^2 : DG^2 :: 2DG : BG$, erect the perpendicular BC to meet the production of OK in C, and from OC cut off CQ equal to BG; the remainder OQ is the measure of the length of the arc OD.

But the rectification of the curve may be performed more directly. For having drawn the tangent OK as before, take BG equal to eight-twenty seventh parts of the parameter AV, and erect the perpendicular BC; or make BT=2DG, and let fall TC perpendicular upon OK.

The Paraboloids appear to have been first proposed in Italy by Cavalleri, who gave their quadrature in his *Exercitationes Mathematicæ*, published in 1647. Roberval and Fermat made similar advances in France. A Dutch mathematician, Van Heuraet, discovered, about the same time, the rectification of the *Semi-Cubic Parabola,* by the ingenious procedure of transforming it into another curve. Fermat, and William Neil, a pupil of Dr. Wallis, arrived at the same conclusion by different methods. But to John Bernoulli we owe the fine discovery, that the evolution of this curve produces the common parabola. His elder brother James, endowed with nearly equal genius, achieved in 1694 the solution of a physical problem proposed by the great Leibnitz, and demonstrated that this Paraboloid has the remarkable property of *paracentric isochronism,* or that it will constrain a body gliding along it to make an uniform vertical descent.

VIII. CATENARY.

If a point starting to the right or the left, gradually bend its course towards a straight line given by position, in such a manner, that the tangent of the angle of deflexion be constantly proportioned to the length of the track; it will describe a curve called the Catenary.

Let AB (fig. 185.) be a horizontal line, to which DCO is perpendicular; and conceive the point C, darting at first parallel to DB or DA, should incessantly deviate from this direction, such, that a given line OC be to CF, the path traced, as radius to the tangent of TFE, the angle of deflexion at F, or the angle which a line touching at F makes with the parallel EF; the curve thus extended on both sides, from C to B and from C to A, will form the *Catenary.*

DEFINITIONS.

1. The horizontal line AB is called the *Base* of the Catenary, and the vertical DC its *Altitude.*
2. The extension DO of the altitude is termed the *Axis*, its extremity the vertex, and the given exterior segment OC the *Parameter.*
3. A line EF parallel to the base and terminating in the curve, is called an *Ordinate*, to which the interior segment CE is an *Absciss.*

PROP. I. PROB.

At a given point in the Catenary to apply a tangent, and find the length of the arc intercepted from the vertex.

Let it be required to measure the arc CF (fig. 185.) of the curve, and draw a straight line that shall touch at its extremity F.

From the vertex, and perpendicular to the axis, let CL be erected equal to the catenarian arc CF, and join OL. By the nature of the curve, OC is to CF or CL, as radius to the tangent of the angle of deflexion EFT, and consequently the angle COL must be equal to EFT. Assume the proximate ordinate *ef*, draw *f*ϕ parallel to *e*E, make L*l* equal to the minute arc F*f*, join O*l*, and having made Oλ equal to it, join *l*λ. The elementary right-angled triangle Fϕ*f* is equal to Lλ*l*, for the hypotenuse F*f* is equal to L*l* by construction, and the oblique angle *f*Fϕ has been proved to be equal to *l*Lλ; whence the side *f*ϕ or E*e* is equal to Lλ. But commencing from the vertex C, the aggregate portions E*e* compose the absciss CE, while the corresponding elementary segments Lλ evidently form the excess of OL above OC; and, therefore, having from the centre O, and with the radius OC, described a circle, CE will be equal to HL, and OE to OL. Join EH, and the triangles COL and HOE, having a common angle LOE contained by equal sides, are equal; whence EHO is a right angle, and EH touches the circle; but the angle HOE being equal to *f*Fϕ, the complementary angle OEH is equal to F*f*ϕ, and therefore EH is parallel to the tangent FT applied to the Catenary at F.

A simple construction is hence indicated. From the centre O with the parameter OC of the curve as a radius describe a circle, to which apply the tangent EH, and complete the parallelogram HEFT; then, from what has been demonstrated, FT touches the curve at F, and must measure the length of the intervening catenarian arc CF.

PROP. II. THEOR.

The ordinates of an equilateral hyperbola, which, with the same vertex as the Catenary, has the parameter for its semiaxis, are equal to the corresponding arcs of this curve.

Let CGI (fig. 186.) be a branch of an equilateral hyperbola constructed on the axis CD, and having O for its centre; any ordinate EG will be equal to the intercepted arc CF of the catenary.

For OS being equal to OC, the square of EG is, from the property of the hyperbola, equivalent to the rectangle under SE and EC the external segments of the diameter CS; but this rectangle is equivalent to the square of the tangent EH to the circle. Wherefore the square of EG is equal to the square of EH, and the ordinate EG itself is equal to EH, that is, to the catenarian arc CF.

Cor. Hence the intervening segment FG is equal to the excess of the arc CF above its ordinate EF. This excess in the case of small arcs is nearly as the square of EC, and consequently the deviation FG augments at first as the square of EC.

PROP. III. THEOR.

The rectangle under an ordinate and the parameter of the Catenary, is equivalent to twice the corresponding hyperbolic sector.

Let CG (fig. 197.) be a portion of the equilateral hyperbola applied to the catenary; join the extremity G with the centre O, and draw the compound ordinate EFG; the rectangle contained by CE and OC is double the space included by the sector COG.

For draw the tangent OH to the circle, let fall HK perpendicular to the axis, and KM perpendicular to the radiant OG, join OH and KG, and assuming the proximate ordinate eg, draw $f\phi$ and $g\gamma$ perpendicular to EG, and gm perpendicular to OG.

From the property of parallel and diverging lines, KM : KD : : gm : $g\gamma$ or $f\phi$; but the triangle HKE is similar to $F\phi f$, and therefore KD : KH : : $f\phi$: $F\phi$; whence, by composition KM : KH : : gm : $F\phi$. Again, the triangle OMK being evidently similar to OEG, KM : OK : : EG or EH : OG; the triangles OKH and OHE are likewise similar, and OK : KH : : OH or OC : EH; whence, by compounding the analogies, KM : KH : : OC : OG, and by identity of ratios OC : OG : : gm : $F\phi$. Wherefore the rectangle under the parameter OC, and the increment $F\phi$ of the ordinate EF, is equivalent to the rectangle under OG and gm, or to double the elementary triangle OgG; but the rectangles under OC and the several portions of the ordinate compose the total rectangle under OC and EF, while the triangles OgG fill up the whole hyperbolic sector COG. The rectangle OC, EF is therefore double of that sectoral space.

PROP. IV. THEOR.

The rectangle contained by the parameter and an ordinate of the Catenary, is equal to the natural logarithm of the ratio of the parameter to the sum of the parameter, the absciss, and the arc of the curve.

Let EQ (fig. 198.) be assumed on the axis equal to the length of the catenarian arc CF; then the parameter OC representing unit, the rectangle OC, EF is expressed by the natural logarithm of the ratio of OC to OQ.

For construct the equilateral hyperbola NCP, having its vertex and centre at C and O; the asymptotes OR and OS will consequently form half right angles on either side of the axis OD. Join QG, and produce it to L, and draw CK and GM parallel to QL and OL.

Because EQ was made equal to CF, it is equal to EG, and therefore QEG is a right-angled isosceles triangle; whence GL and CK are both perpendicular to the asymptote OS. But, from the property of the hyperbola, the rectangular space KCGL is equivalent to the sector COG, and is the natural logarithm of the ratio of OK to OL, when OK or CK represents unit. That ratio is evidently the same as ratio of OC to OQ; but an asymptotic space similar to KC.GL, and having OC substituted for CK, would be augmented in the duplicate ratio of OC to OK, and would consequently be doubled. Wherefore, the rectangle OC,EF which is double the sector COG, is expressed numerically by the logarithm of the ratio of OC to OQ, when OC represents unit.

Cor. From the property of parallel and diverging lines, OM is to OC as GL to CK, that is, from the property of the asymptote, as OK to OL or OC to OQ; wherefore OM : OQ :: $OC^2 : OQ^2$, or OC : OQ :: $\sqrt{OM} : \sqrt{OQ}$, and the rectangle OC,EF is hence equivalent to half the natural logarithm of the ratio of OM to OQ. This logarithm might consequently be represented by half the interval on the logarithmic curve (fig. 169.) between ordinates equal to OM and OQ. The ordinates LN or OM, OC, and OQ or OH must thus be equidistant from OC; whence the rectangle OC,OI is equal to OC,EF, and the segment OI is therefore equal to the ordinate EF of the Catenary. Again, since MGQ is a right-angled isosceles triangle, EM is equal to EQ, and consequently FG is equal to FH.

Hence an easy practical method of describing the Catenary. Construct (fig. 188.) a logarithmic curve LCH, having its subtangent equal to the ordinate OC, and form an equal and similar curve in an opposite direction GCK, draw any compound ordinate IGH to both curves, and bisect the difference GH in F, which will be a point in the Catenary. A multitude of points so found will mark the trace of that curve.

PROP. V. THEOR.

The exterior catenarian space is equivalent to the rectangle under the parameter, and the excess of the length of the curve above the ordinate.

If the rectangle ECFL (fig. 186.) be completed, the space LCF intercepted between it and the curve will be equivalent to the rectangle under OC the parameter and the excess of the catenarian arc CF above its ordinate EF.

For, with the parameter as a radius describe the circle SHC, and construct the corresponding equilateral hyperbola. It is evident, that the element Ff of the curve is equal to $G\gamma$, the variation of the hyperbolic ordinate; but since the elementary triangle $F\phi f$ is similar to EHO, Ff or $G\gamma : F\phi :: OE : OH$ or OC, and consequently $G\gamma - F\phi : F\phi :: CE : OC$; wherefore $CE.F\phi = OC\,(G\gamma - F\phi)$. But the aggregate of the elementary rectangles $CE.F\phi$ or $FLl\phi$ forms the exterior space LCF, and the collected rectangles under OC and the difference of $G\gamma$ and $F\phi$ compose the rectangle under OC and the excess of EG or CF above EF; the exterior space of the catenary is therefore equivalent to the rectangle under OC and FG, or under OC and the excess of the arc CF above its ordinate.

Cor. 1. Hence the catenarian space ECF is equivalent to the excess of the rectangle OE,EF above OC,CF. For the exterior space $LCF = OC.FG = OC.CF - OC.EF$, and consequently the interior space $ECF = EC.EF + OC.EF - OC.CF = OE.EF - OC.CF$.

Cor. 2. Considering the curve as approaching to a parabola, the exterior space $LCF = \frac{CE.EF}{3}$; whence $OC.3FG = CE.EF$, and $OC : EC :: EF : 3FG$ nearly.

PROP. VI. THEOR.

The radius of curvature corresponding to any point of the Catenary, is a third proportional to its parameter, and the line compounded of the parameter and the absciss.

If there be drawn from the point F (fig. 187.) of the Catenary a perpendicular to the tangent, or a normal FQ, and from the proximate point f another normal fQ, they will meet in Q, the centre of an osculating circle, and OC : OE : : OE : FQ.

Parallel to those normals, draw OL and Ol to meet CL, the perpendicular to the axis at the vertex C, and from the centre O describe the minute arc Lλ.

The right-angled triangle OCL is obviously similar to the elementary triangle Lλl, and consequently OC : OL or OE : : Lλ : Ll or Ff. But the very acute triangles FQf and LOL being likewise similar, Lλ : Ff : : Oλ or OE : FQ; whence OC : OE : : OE : FQ.

Cor. 1. If LR be drawn perpendicular to OL, it will intercept a portion of the axis OR equal to FQ, the radius of curvature.

Cor. 2. Since OLR is a right-angled triangle, OC : CL : : CL : CR; whence the excess CR of the radius of curvature above the parameter OC, is a third proportional to this parameter and the length of the intercepted arc CF.

PROP. VII. THEOR.

Every Catenary is a portion of a similar curve.

If ACB and KLM (fig. 189.) be any two Catenaries, they belong to similar curves.

For make the absciss CD to the absciss LN as the parameter OC to the parameter PL, draw the ordinates DH and NM and the proximate ordinate *dh* and *nm*, and apply the tangents DH and NQ to the parametric circles.

Since OC : CD : : PL : LN, by composition OC or OH : OD : : PL or PQ : PN, and consequently the right-angled triangles OHD and PQN are similar; but these triangles are likewise similar to the elementary triangles *b*ρB and *m*μM, for DH and NQ are parallel to the minute segments B*b* and M*m* of the curves. Whence the aggregate sides *b*ρ, ρB, B*b*, and *m*μ, μM and M*m*, that is, the abscissæ CD and LN, the ordinates DB and NM, and the branches CB and LM of these curves, are all proportioned to the parameters OC and PL. The parameter is thus an index to the Catenary, as it is to the parabola, both of which may be prolonged to any extent.

The *Catenary* has its name from *Catena, a chain*; being the curve which a regular and very flexible chain will assume, if suspended loosely from both ends. It seems to have been first noticed by the famous Galileo, who proposed it as the figure of an arch of equilibration, but unfortunately mistook it for a Parabola. In fact, the Catenary, near its vertex, differs insensibly from that curve, but afterwards deviates more considerably. The tangent of the angle of deflection, which in the Catenary is proportional to the length of incurvation, may be shown to be proportional to the ordinate in the Parabola. But the ordinate commences in a ratio of equality to the curve, and then contracts with increasing rapidity; hence the Parabola diverges faster from its axis than the Catenary.

The error of Galileo, in confounding those two curves, was not perceived till Joachim Jungius, in 1669, ascertained, by actual experiment, that the Catenary is neither a

Parabola nor an Hyperbola. It was in 1691, that the penetrating genius of James Bernoulli discovered the true nature of the catenarian curve. A similar investigation was soon produced by John Bernoulli, by Huygens, and by Leibnitz. This latter philosopher, whose powers of invention and stores of learning were alike transcendant, discovered the fine relation of the Catenary to the Logarithmic Curve.

The most difficult properties of the Catenary were revealed before the close of the seventeenth century. This curve is entitled to particular attention, not only because it throws light on the theory of arches, but because it applies directly to the construction of suspended bridges, which are now deservedly coming into repute.

X. TRACTORY.

If a given straight line have one end drawn along an extended line given by position, the other end, following its path or direction, will trace out a curve, called the Tractrix *or* Tractory.

Let PT (fig. 190. and 191.) be a straight line of a determinate length, of which the end T is drawn along the straight line SOT given by position; the other end P, constrained to pursue the same track, will describe a Tractory.

The nature of the curve consists in having every tangent PT of the same length. The Tractory must hence include four branches which stretch indefinitely either way on both sides of the directing line, and rise into two cusps at their conjunction, when the tangents become vertical.

DEFINITIONS.

1. The extended straight line SOT on which the tangents rest, is called the *Axis.*
2. When the tangent acquires a vertical position OC, it is termed the *Radius,* its extremity O being the centre.
3. A circle described from the centre of the curve through the cusp C is called the *Generating Circle.*

PROP. I. THEOR.

The subtangent of the Tractory is equal to the corresponding ordinate of the generating circle.

From the point P in the curve (fig. 190.) draw PN perpendicular to the axis, and PKH parallel to it and meeting the circumference of the generating circle; then wil KH be equal to the subtangent NT.

For join OH, and the right-angled triangles OKH and PNT, having the hypotenuse OH equal to OC or PT, and the side OK equal to the opposite side PN of the parallelogram OKPN are equal, (I. 21. Geom.) and consequently the base KH, or the ordinate to the circle, is equal to the subtangent NT.

Cor. 1. Hence the compound line PKH is equal to ONT; for KH being equal to NT, add to each of them the equal segments PK and ON, and the sum PH is equal to OT.

Cor. 2. Hence the tangent PT of the tractory is always parallel to the corresponding radius OH of the generating circle. For the angles KOH and NPT being equal, and the sides OK and PN parallel, the other sides OH and PT must be likewise parallel (I. 29. Geom.)

PROP. II. THEOR.

The space included between the radius, the axis, a portion of the curve, and a tangent of the Tractory, is equal to the area of the corresponding sector of the generating circle.

Having applied a tangent PT (fig. 191.), and drawn PKH parallel to the axis of the curve; the mixtilineal space OCPT is equivalent to the circular sector COH.

For assume the proximate point p, draw the tangent pt and the parallel pkh, and join OH and Oh. Since the radii OH and Oh are equal and parallel to the tangents

PT and *pt*, the elementary triangle HO*h* may be considered as equal to TP*t* or T*pt*; wherefore the amount of all the triangles HO*h* which form the sector COH is equivalent to the aggregate of the triangles T*p*C which compose the mixed space OCPT.

Cor. Hence the quadrant COR is equivalent to the indefinite space included between the radius, the asymptote and the extending curve. Hence also the whole space comprehended by the four branches of the tractory is equivalent to the generating circle.

PROP. III. THEOR.

The segment of the axis intercepted between the centre and a line touching the Tractory, is expressed by the natural logarithm of the ratio of the radius to the cotangent of half the complement of the corresponding arc of the generating circle.

Assuming the radius OC (fig. 191.) as unit, any portion OT of the axis will be denoted by the natural logarithm of the ratio of radius to the cotangent of half the complement of the arc CH, or the tangent of half the compound arc SCH.

For join S, where the circle cuts the axis, with the proximate points H and *h*. The elementary triangles HS*h* and *i*SI are similar, since they have the same vertical angle, and the angle H*h*S, standing on the compound arc SCH equal to *i*IS, which is subtended by the arcs CH and SC′ or SC; whence SH : S*i* or SI : : H*h* : I*i*; but SH : SI : : OK : OI, and consequently OK : OI : :

H*h* : I*i*, or alternately OK : H*h* : : OI : I*i*. Again, the minute elementary triangles H*lh*, *p*πB, and *t*τT are all evidently similar to HKO, and therefore similar to each other; whence H*h* : *hl* or Pπ : : T*t* : Tτ or P*p*, and alternately H*h* : T*t* : : Pπ : P*p*; but Pπ : P*p* : : OK : OH, and hence H*h* : T*t* : : OK : OH, or OK : H*h* : : OH : T*t*. Now since OK : H*h* : : OI : I*i*, consequently OI : I*i* : : OH or OC : T*t*, or alternately OI : OC : : I*i* : T*t*. Wherefore OI and O*i*, if planted at the points T and *t*, would be proximate ordinates of a logarithmic curve having OC for its subtangent, and hence the absciss OT is equal to the natural logarithm of the ratio of OI to OC. But OI is the tangent of the angle RSH at the circumference of the circle, and consequently the tangent of half the arc RH, the complement of CH. Again, the radius being ame an proportional between the tangent and the cotangent, the ratio of OI to OC is the same as the ratio of radius to the cotangent of half the complement of CH. The proposition is therefore established.

Cor. Since OK : OC : : H*h* : T*t*; the increments of the segment OT are always inversely as the cosine of the arc CH; wherefore, if that arc represent the Latitude, the corresponding line OT will exhibit the Meridional parts, Mercator's Projection.

PROP. IV. THEOR.

The Tractory is the involute of the Catenary, which has the radius for its parameter.

If a thread attached at some distant point B (fig. 190.) of a Catenary, and bent along the convexity of the curve to the vertex C, be then gradually unfolded, its expanding end will trace a Tractory, whose radius is the parameter OC.

Let the radius of curvature come into the position FP, draw FT perpendicular and FE parallel to ROS, apply the tangent EH to the generating circle, and join FH and PT.

From the property of the Catenary, the straight line EH which touches the circle is equal to the arc FC of the curve, and is parallel to its tangent FP; but the radius of curvature FP is equal to the arc FC, from which it has been unfolded, and is therefore equal to EH; whence EP and HF, which join those equal and parallel lines, are themselves equal and parallel. Wherefore OT, being likewise equal and parallel to EP, the figure OEPS is a parallelogram whose opposite sides PT and OE are equal and parallel; but since FP and PT are parallel to EH and EO, the contained angle FPT is equal to HEO, and is therefore a right angle; whence the line PT, being perpendicular to the radiant FP, must coincide with minute portion of the circle or element of the curve at P, and this tangent is equal to OH or OC the parameter of the Catenary. But such is the genesis of the Tractory, which is hence the involute of the catenary.

Scholium. A Tractory being traced mechanically, the series of points may be thence readily found in the Catenary; for erect any perpendicular TF to ROS, inflect TP to the curve equal to OC, and at right angles to it draw PF to meet TF in F, which must evidently be a point in the catenary.

An instrument might therefore be constructed that shall combine the mechanical construction of the Tractory and of the Catenary. Let a thin bar of wood or brass SG (fig. 220.) about two feet long, with a groove running along the middle, have a perpendicular arm OT projecting about four and a half inches each way; to this frame is annexed a similar piece MPSPF, of which the single arm PS is eight inches long, with holes about half an inch apart drilled from the middle to S, having a very small sharp-edged wheel at P, turning about an axis in the same plane, but at right angles to PS; one of those holes being passed over a steel pin fixed at S, in the direction of the groove GS, and a fine pencil secured by a slider being inserted in the concourse of both grooves at R, the instrument is fit for action. By sliding OST along the side of a drawing board, and pressing the small wheel at P, it will describe a Tractory, while the pencil carried in the intersection of the perpendiculars PF and SG will trace the Catenary.

Such an instrument, with proportionally shorter arms, would be useful to Architects, by enabling them to sketch festoons correctly.

The *Tractory* was first proposed by the celebrated Huygens. It may be considered as derivative from the Catenary, of which it is the involute. John Perks, in England, likewise demonstrated its leading properties, under the appellation of the *Equitangential Curve*. John Bernoulli discovered that it is *Tautochronous* in a medium resisting as the squares of the velocity, or that, in such a medium, bodies would effect their descent from any point of this curve in the very same portion of time.

The relation of the Tractory to the Meridional Projection of the Globe is very remarkable, while the facility of tracing the curve mechanically brings it closer under our observation.

XI. MAGNETIC CURVE.

If the radiants from two given poles revolve with angular velocities proportional to their varying lengths, the point of mutual intersection will trace the MAGNETIC CURVE.

Let AC and BC (fig. 199 and 200.) be two radiants which turn about the poles A and B, and let them in the same instant move into the proximate positions A*c* and B*c*; if the angle CA*c* be to CB*c* as AC to BC, the points C and *c* will occupy a *Magnetic Curve.*

DEFINITIONS.

1. The straight line which joins the two poles is called the *Base* of the curve, and its extension the *Axis.*
2. The angles BAC and ABC which the radiants make with the base are called the *Polar Angles.*

It is obvious, that the Magnetic Curve will assume two distinct forms according as its radiants turn in the same or in opposite directions. In the latter case, both the polar angles augmenting at the same time, the curve will consist of a single concave arc; but in the former case, the angles from the one pole increasing, while those from the other are diminishing, the curve will divide and spread itself into two branches. The one species of curve is hence *Convergent*, as in fig. 199, and the other *Divergent*, as in fig. 200.

PROP. I. THEOR.

The sines of the angles which a tangent forms in the Magnetic Curve with the radiants drawn to the point of contact, are proportional to the squares of those lines.

Let TC (fig. 199 and 200.) touch the curve at the point C, and make the angles TCA and TCB with the corresponding radiants; then will $\sin TCA : \sin TCB :: AC^2 : BC^2$.

For assume the proximate point c, to which inflect the radiants Ac and Bc, make BD equal to AC, and from A and B describe the minute arcs $c\alpha$, $C\beta$, and $D\delta$.

The arcs $c\alpha$ and $D\delta$ are obviously the measures of the elementary angles CAc and CBc, and consequently, from the genesis of the curve, $c\alpha : D\delta :: AC : BC$; but, since the sectors $CB\beta$ and $DB\delta$ are similar, $D\delta : C\beta ::$ BD or $AC : BC$; wherefore, by composition of ratios, $c\alpha : C\beta :: AC^2 : BC^2$. Again the right-angled elementary triangles $c\alpha a$ and $C\beta a$ having a common vertical angle, are similar, and $c\alpha : ac :: C\beta : Ca$, or alternately, $c\alpha : C\beta :: ac : Ca$; whence, by identity of ratios, $ac : Ca :: AC^2 : BC^2$. Now, in the triangle Cac, $ac : Ca ::$ $\sin cCa$ or $\sin TCA : \sin Cca$ or $\sin TCB$; wherefore, $\sin TCA : \sin TCB :: AC^2 : BC^2$.

Cor. 1. Hence a tangent cuts the extended base into segments which are as the third powers of the corresponding radiants. For draw AE parallel to BC, and the triangle EAC being similar to caC, $AE : AC :: ca : Ca ::$ $AC^2 \; BC^2$; wherefore, $AE : BC :: AC^3 : BC^3$; but, from the property of parallel and diverging lines, AE : BC : : AT : BT, and consequently AT : BT : : $AC^3 : BC^3$. In fig. 199 and 200, the radiants AC and BC are as two to three; whence TA : TB : : 8 : 27. Wherefore, AT in fig. 199. is 8-19 parts of AB; but in fig. 200, it is equal only to 8-35 parts of the same distance.

Cor. 2. Hence, if tangents be drawn from any point in the axis to different magnetic curves constituted upon the same base, the points of contact will lie in the circumference of a given circle. For T being a given point, the ratio of AT to BT, or that of AC to BC is given, and consequently the ratio of AC to BC is given, and the locus of C (Geom. Anal. III. 13.) is a given circle. In the illustrative figures 201 and 202, where the radiants AC and BC are as two to three, the distance AX of the circle of contact is two-fifths of AB, and its radius six-fifths. The correspondence with theory in these delineations is sufficiently striking.

Scholium. The method of applying a tangent to the Magnetic Curve is hence easily perceived. In the *Convergent* Curve, the tangent CT lies beyond the nearer pole; but, in the *Divergent* Curve, CT falls within the poles. The former species of curve is therefore always concave towards its base, while the latter is convex.

Since C*a* : *ca* : : $BC^3 : AC^2$, the little spaces C*a* and *ca* in the direction of the poles A and B, are inversely as the squares of the distances CA and CB. These spaces may therefore represent the forces of magnetic attraction or repulsion emanating from A and B. An attraction C*a* combined with a repulsion *ac* would produce the oblique direction C*c* (fig. 199.); while the same attractive force conjoined with another attraction *ca'* would give the result C*c* (fig. 200.). Hence very short needles or iron-filings, subject to those attractive or repulsive powers, but allowed to arrange themselves freely, will coalesce into the elementary portions of the curve. If A and B be dissimilar poles, the curve, being composed of projecting portions Cc, will exhibit the convergent form; but if these poles be similar, the curve, consisting of minute reclined portions, will become divergent.

PROP. II. THEOR.

The sum of the cosines of the polar angles in the Convergent Magnetic Curve, but their difference in the Divergent form of the curve, remains constant.

The cosines of the angles BAC and ABC (fig. 199.) compose, by addition, a certain invariable amount at every point of the curve; but in fig. 200. the cosines of the polar angles have always the same difference throughout each branch of the curve.

For, let fall upon the axis the perpendiculars $c\varphi F$ and DdG, and draw the parallels $\alpha\varphi$ and δd. The elementary triangle $c\varphi\alpha$ is evidently similar to the right-angled triangle AFc, and $\alpha c : \alpha\varphi :: Ac$ or $AC : cF$; but the triangles $Dd\delta$ and BFc being likewise similar, $D\delta : \delta d ::$ BC or $Bc : cF$. By alternating both analogies, $\alpha c : AC ::$ $\alpha\varphi : cF$, and $D\delta : BC :: \delta d : CF$; but αc and $D\delta$, being the measures of the variation of the polar angles BAC and ABC, are as the radiant AC to BC, consequently $\alpha\varphi : cF$ $:: \delta d : cF$, and $\alpha\varphi = \delta d$, or $fF = Gg$. Now, fF (fig. 199.) is the diminution which the cosine AF suffers in the movement of the radiant AC from the point C to c, and the equal portion δd is the equal increase which cosine BG receives in the corresponding advance of the radiant BC. Wherefore, the cosines AF and BG must maintain the same amount throughout the Convergent Curve.

In the Divergent Curve (fig. 200.) $\alpha\varphi'$ lies on the other side of the perpendicular cF, the cosines of the polar angles BAC and ABC both receive equal increments $f'F'$ and $G'g'$, as the radiants expand from c to C, and, consequently, those cosines AF' and BG' must have always the same difference.

Scholium. It is hence easy to trace, through any point C, a Magnetic Curve whose poles are given. From A and BAO, with a radius (fig. 199 and 200.) equal to half their distance, describe two semicircles, draw AC and BC to cut the circumference in the points H and I, from which let fall the perpendiculars HK and IL, take any segment KQ and make LR equal to it, either on the same or on the opposite side according as the curve is Convergent or Divergent, erect the perpendiculars QM and RN to meet the circles, and inflect the radiants AMP and BRN, whose intersection P will be a point in the curve. But if Lr (fig. 200.) be made equal to OQ, and the perpendicular rn erected, the radiant Bn will, by its intersection with the radiant AM, assign the point p in the other branch of the curve.

When the radiant BC coincides with the axis, the radiant AC will become a tangent to the curve, and will consequently mark the direction of its origin. To find this, make KS equal to LO, either on the same side as in fig. 199, or on the opposite side as in fig. 200; erect the perpendiculars SV to meet the circle, and SV will be the radiating tangent.

But a system of Magnetic Curves may be constructed more expeditiously. Divide the radius of the circles into any even number of equal parts, suppose twelve, erect ordinates from all the points of extended section, and draw radiants through their summits. Suppose the radiants of the pole A (fig. 199.) to be reckoned from X to Y, and those of B from X to Z. For the Convergent Curve, the intersection of the radiant 1 from the pole A, with the radiant 1 from the pole B, will assign the limit of the curve next the axis; the intersection of the radiants 2 and 1 from A with the radiants 1 and 2 from B will mark two points in the next curve; the intersection of the radiants 3, 2, and 1 from A with the radiants 1, 2, and 3 from B will give three points in the curve beyond it; and the intersection of the radiants 4, 3, 2, and 1 from A, with the radiants 1, 2, 3, and 4 from B, will trace another exterior curve. In this way, the process may be pursued through the whole series of curves, the corresponding initial and final tangents being the succeeding radiants 2, 3, 4, 5, &c. from the poles A and B.

For the Divergent Curve, the intersections of the radiants 1, 2, 3, 4, 5, &c. from the pole B, with the radiants 2, 3, 4, 5, 6, &c. from the pole A, will mark the first branch from O towards Y; the intersections of the same radiants with the radiants 3, 4, 5, 6, 7, &c. will trace the next branch; and the intersections of still the same radiants with the radiants 4, 5, 6, 7, 8, &c. will assign the third branch. This procedure is easily carried on, and a similar mode will assign the opposite branches. The radiants 1, 2, 3, 4, 5, &c. from A will, by their intersections with the radiants 2, 3, 4, 5, 6, &c. from B, with the radiants 3, 4, 5, 6, 7, &c., and with radiants 4, 5, 6, 7, 8, &c. will trace out the successive diverging curves from O to Z.

The tangents to all these curves at A and B are the preceding radiants 0, 1, 2, 3, &c.

The figures 201 and 202 were reduced from curves actually delineated in this way, omitting only, for the sake of distinctness, some of the initial and intermediate curves.

The Magnetic Curve is formed experimentally by strewing fine iron-filings or very short bits of iron wire on a smooth sheet of paper, or on a thin plate of glass laid over a magnetic bar, or over the interval between the similar poles of two opposite bars. By a slight agitation to favour their mutual arrangement, those minute ferruginous fragments will dispose themselves into the Convergent or Divergent Curves now investigated.

These curves have been regarded with wonder by a certain class of dreaming philosophers, who did not hesitate to consider them as the actual traces of an invisible fluid, perpetually circulating between the poles of the magnet. Such a crude notion scarcely deserves a moment's notice; but the entire conformity between theory and experiment in the representation of the Magnetic Curves corroborates the results of other observations, and proves, that the forces of magnetic attraction and repulsion follow, in their graduating intensities, the great law of gravitation, and are inversely as the squares of the distances of the poles or centres of action.

XII. CURVE OF SINES.

If the arcs of a circle be assumed as Abscissæ, to which the corresponding sines are ordinates, the summits of these will mark out the Curve *of* Sines.

Let the arcs of a circle, of which O (fig. 203.) is the centre, and OC the radius, be extended along the axis AB, and perpendiculars, erected equal to the corresponding sines, will exhibit a line called the *Curve of Sines.*

This curve will evidently rise to an elevation OL equal to the radius of the generating circle, then descend from its vertex till it cross the axis at a distance OB, equal to OA beyond the centre, and form a similar curve below; it will next reappear above the axis, and continue to repeat the same train of convolutions.

PROP. I. PROB.

To apply a tangent at any point of the Curve of Sines.

Let it be required to draw a line touching this curve in the point F (fig. 203.)

Assume the proximate point *f*, and draw FGI and *fgi* parallel, and FE, GH, *fφh* and *gγ* perpendicular to the axis.

The elementary triangle $f\varphi F$ being similar to FET, $f\varphi : \varphi F :: FE : ET$; but, from the genesis of the curve, the minute segment φF is equal to the elementary arc Gg of the circle, and consequently $f\varphi$ or $g\gamma : Gg :: FE : ET$. Now the elementary triangle $g\gamma G$ is similar to OHG, and $g\gamma : Gg :: OH : OG$; wherefore $FE : ET :: EH : OG$, and the subtangent ET is hence given.

A simple construction results. Make OK equal to OH, join CK, and parallel to it draw FT, which will be the tangent required. For the triangle TFE is obviously similar to KOC, and consequently $FE : ET :: OK$ or $OH : OC$ or OG.

Cor. At the point A, the cosine OK or OH is equal to the radius OL, and hence the curve at its origin crosses the axis at the inclination of half a right angle.

PROP. II. THEOR.

The area of any segment of the Curve of Sines is equivalent to the rectangle under the radius of the generating circle, and the corresponding versed sine.

The space AEF (fig. 203,) inclosed by the portion of the curve AF, and its absciss and ordinate AE and EF, is equivalent to the rectangle under the radius OL and the versed sine CH.

For the elementary triangles $G\gamma g$ being similar to GHO, Gg or $F\varphi : G\gamma :: OG$ or $OL : GH$ or FE, and consequently $F\varphi . FE = G\gamma . OL$, wherefore the aggregate of the rectangles $F\varphi , FE$ or of $F\varphi e E$ is equal to the accumulated rectangle, $G\gamma$.OL. But the elementary spaces $F\phi eE$ compose the segment AEF of the curve, and the rectangles under the radius and the increments $G\gamma$ make up the rectangle under OL and the versed sine CH. Wherefore the area of the segment AEF is equivalent to the rectangle under OL and CH.

Cor. Hence the area of half the curve AOL or OLB is equivalent to the square of the radius OL or OLPD, and the area of the whole curve is equivalent to the rectangle under the radius OL and the diameter CD.

The *Curve of Cosines* is only a slight modification of the *Curve of Sines.* Take LR equal to the circular arc LV, and make the ordinate RS equal to the cosine OI, and the Curve of Sines will be converted into a Curve of Cosines, commencing at the vertex L, and redoubling along the extended axis OB,

If the Curve of Sines have its ordinates diminished or augmented in the same ratio, it will be changed into the *Harmonic Curve.* This mutation is represented in fig. 204. where the ordinates of the Curve of Sines are augmented by one-third, and likewise subdivided into three equal portions. It is evident, that the areas of segments of these curves will be changed in the same ratio; and it will also follow, that tangents drawn from any point in the axis to the several curves will make their contacts in a straight line perpendicular to the axis.

XIII. CURVE OF TANGENTS.

If, from a given point in an extended line, segments be taken equal to the arcs of a given circle, and perpendiculars erected equal to the corresponding Tangents, their vertices will be in what is thence called the CURVE *of* TANGENTS.

On the extended line AB (fig. 192.) let any portion AE be taken equal to the arc CG of a given circle, and from the point of section E draw the perpendicular EF equal to the tangent CT; the locus of point F will be the *Curve of Tangents.*

DEFINITIONS.

1. The given circle is termed the *Generating Circle.*
2. The extended line AB is called the *Axis*, any segment AE being an *Absciss*, and the perpendicular EF its *Ordinate*.

This curve commencing at A, must evidently, in the first quadrant, shoot indefinitely along the middle ordinate OM; but, on the second quadrant, it will re-appear on the opposite side, bending gradually from the ordinate ON till it reach B at a distance beyond O equal to AO; it will then stretch obliquely upwards, forming a branch similar to the portion at its origin, and will afterwards repeat the same series of unlimited arcs.

PROP. I. PROB.

To draw a straight line that shall touch at a given point of the Curve of Tangents.

Let it be required to apply a tangent at the point F fig. 192.) of the curve. Assume the proximate point *f*, and draw the secant *f*FS; draw also Fφ and *f*φ parallel to AO and OM, make the minute arc G*g* equal to Fφ, extend O*g* to meet CT in *t*, and erect T*t* perpendicular to OT.

The elementary triangle *t*Tτ being similar to TOC, it follows, that T*t* : Tτ :: TO : OC; and CP being drawn perpendicular from the right angle C, TO : OC :: TC : CP, and consequently T*t* : Tτ :: TC : CP. Again, from the property of parallel and diverging lines, Tτ : G*g* :: TO : GO or OC; but PQ being let fall perpendicular to AO, TO : OC :: CP : PQ. Wherefore, by composition of ratios, T*t* : G*g* :: TC : PQ, the elementary triangle Fφ*f* being similar to SEF, *f*φ or *t*T : Fφ or G*g* :: FE or TC : ES, and consequently TC : PQ :: TC : ES, and the subtangent ES is equal to the assigned part PQ.

PROP. II. PROB.

To find the area of any portion of the Curve of Tangents.

Let it be required to measure the space AFE (fig. 192.) included between the curve, its axis, and an ordinate FE.

Since the elementary triangle $gG\gamma$ is similar to OTC $G\gamma : Gg :: CT : OT$, and $OT : G\gamma = CT.Gg$. But the minute arc Gg is equal to the difference $F\varphi$ of the proximate *abscissæ*, and CT is equal to the ordinate EF; wherefore the rectangle $OT, G\gamma$ is equivalent to $EF, F\varphi$ the element of the curved space. Extend the perpendicular GH on the opposite side till HL be equal to OT, and draw the parallel $g\gamma hl$; the rectangle $HLlh$ is hence equivalent to the elementary rectangle $GHh\gamma$ of the curve, and therefore the space KCHL is equivalent to AEF. But OT or HL : GO or OC :: OC : OH, and $HL.OH = OC^2$; whence the curve KL is a rectangular hyperbola, of which OA and ON are the asymptotes. This space KCHL, the measure of the curvilineal area AEF, is thus given.

Cor. Let the base AB be denoted by $2b$, the radius OC by r, and the absciss AE by x; then the ordinate EF $= r\ tan.\frac{x}{r}$. Again, the curvilineal space AEF, being equivalent to the hyperbolic surface KCHL, is expressed by $r\ log.\left(\frac{b-x}{r}\right)$.

XIV. CURVE OF SECANTS.

If, from a given point in an extended line, segments be taken equal to the arcs of a given circle, and perpendiculars erected equal to the corresponding Secants, their vertices will lie in the Curve of Secants.

On the extended line AB (fig. 193), let the segment AE be taken equal to the arc CG of a given circle, and from the point E draw the perpendicular EF equal to the *Secant* OT; the locus of F will be the *Curve of Secants.*

This curve will evidently begin at I, the top of a perpendicular AI, equal to the radius; it will then bend gradually upwards till it vanishes into space along the middle perpendicular OM; but it will next form a similar branch on the other side of OM, descending from indefinite remoteness till it falls into K, corresponding to the point I. These two branches resting on the axis will be repeated through all the successive semicircles.

DEFINITIONS.

1. The given circle is termed the *Generating Circle.*
2. The extended line AB is called the *Axis*, of which any segment AE is an *Absciss*, and the perpendicular EF an *Ordinate.*

PROP. I. PROB.

At any point in the curve of secants to apply a tangent.

Let it be required, through the point F, (fig. 193.) to draw a straight line that shall touch the curve.

Through the proximate point f extend the line of fFS, draw Fφ and $f\varphi$ parallel to AO and OM, make the elementary arc Gg equal to Fφ, project the secant O$g\tau t$, and erect Tτ perpendicular to OT.

It is obvious, that $f\varphi$ or $t\tau$: Fφ or Gg : : EF or OT : SE; but, having drawn OR perpendicular to OT and meeting TC, $t\tau$: Tt : : OT : OR, and Tt : Gg : : OT : OG or OC : : OR : CR; whence, by compounding these analogies, $t\tau$: Gg : : OT : CR; and consequently OT : CR : : OT : SE, and CR is equal to the subtangent SE.

PROP. II. PROB.

To find the area of any portion of the Curve of Secants.

Let it be required to measure the area of the space intervening between the origin of the curve (fig. 193.), and the ordinate corresponding to the secant OT.

Draw GN to the extremity of the perpendicular diameter MN, make OL : OH : : OT : LP; assume likewise the proximate secant Ot, join gN, and make Ol : OH : : Ot : lp.

Considering the elementary arc Gg as coincident with the tangent at G, the triangles GNg and LNl are similar; for, having the same vertical angle, the angle NgG standing on the arc GCN is equal to the angle NLl subtended by the two arcs CG and ND, which are together equal to GCN. Wherefore NL : Ng or NG : : Ll : Gg; but, from the property of parallel and diverging lines, NL : NG : : OL : OH, and consequently, OL : OH : : Ll : Gg. Now, by construction, OL : OH : : OT : LP, and by identity of ratios OT : LP : : Ll : Gg; whence OT.Gg=LP.Ll. But the rectangle OT.Gg is an element of the curve, and therefore LP.Ll, or the minute space PLlp, is likewise such an element. Again, from the property of parallel and converging lines, OH : OG or OC : : OC : OT, and OH.OT$=$OC^2; but since OL : OH : : OT : LP, it follows, that the rectangle OL.LP is equivalent to OH.OT, and therefore to the square of OC or CK. Wherefore the curve KP is a rectangular or equilateral hyperbola, which has O for its centre, and OC and ON for its asymptotes. The area of a portion of the curve of secants is hence measured by the hyperbolic space KCLP.

Scholium. It is evident that OL, the absciss of the hyperbola, is the tangent of the angle ONL or of half the complement of the arc CG. Whence, if CG be denoted by a, OC by r, and the ratio of the diameter to the circumference of a circle by $1 : \pi$; the area of the curve of secants corresponding to the arc CG will be expressed by $r^2\ log.\ tan.\ \frac{\pi r - 2a}{4r}$. If a were to mark the length of the arc in degrees, the radius being reckoned unit; the expression for the curvilineal area would become simply *log. tan.* $\frac{90°+a}{2}$, the known measure of the meridional extension according to Wright's or Mercator's projection of the sphere. In that delineation, where the several meridians are represented by parallel lines, while the relative proportions of the degrees of latitude and longitude are preserved, all the vertical divisions on those meridians are enlarged in the inverse ratio of the cosine of the latitude, or in the direct ratio of its secant. The length of this artificial meridian is therefore as the area of the curve of secants, and is hence denoted by the logarithmic tangent of an arc composed of 45 degrees and half the given arc. The same conclusion is derived from the property of the Tractory.

The Curves or Figures of Sines, Tangents and Secants appear to have been suggested by the mode which Edward Wright took to form the Meridional Parts, by continually adding the successive secants of every minute of the quadrant. The extension of the meridian by his projection must evidently be proportional to the area of the curve of secants; but the beautiful property that this area is expressed by the logarithm of the cotangent of the semi-arc was first discovered by our famous countryman James Gregory, and demonstrated in his *Exercitationes Geometricæ*, published at London in 1668.

These curves had attracted the notice of Mathematicians early in the seventeenth century, and their general properties were gradually brought into view. If, instead of the area themselves, their complements or supplements be assumed, the resulting figures will continue essentially the same, only affected by slight changes in the ramification. In the curve of sines, the successive branches are formed equally on both sides of the axis; in the curve of cosines, a whole branch lies on the one side and two half branches occupy the other side of the axis.

The Companion of the Cycloid is fundamentally the same as the *Curve of Sines*. For through the centre of the generating circle draw (fig. 176.) AOC parallel to the base, and let GK be any other parallel; then AO is equal to the quadrant BE, and GK is equal to the arc BI; consequently AF is equal to the arc IE, and the perpendicular GF is equal to IH its sine. Wherefore, on the extended axis AOC, the *abscissæ* being taken equal to the arcs of the generating circle, the corresponding ordinates are equal to their *sines*.

The very ingenious Dr Brook Taylor discovered that the *Harmonic Curve*, or the curve which a musical string assumes in the act of vibration, is only a prolonged curve of sines, having all its ordinates diminished after the same varying ratio. This beautiful result, he demonstrated in his profound work on *Increments*, which appeared in 1716.

XV. EQUABLE SPIRAL.

If a straight line revolve uniformly about its extremity, a point which travels likewise uniformly along it, will describe the EQUABLE SPIRAL.

Let (fig. 209.) the straight line PABCD turn always uniformly about its extremity P; a point which advances along it likewise by an uniform motion and arrives successively at A, B, C, D, &c., and at *a*, *b*, *c*, *d*, &c., will trace out the curve called the Equable Spiral.

DEFINITIONS.

1. The point P about which the line revolves is called the *Pole*.
2. The portion of the revolving line intercepted between the pole and the curve is termed a *Radiant*.
3. The primary position of the revolving line forms the *Axis*.

PROP. I. THEOR.

The radiant of the Equable Spiral is proportional to the angle which it has described.

For while the revolving line (fig. 210.) passes into the position PB, and measures out the angle APB, the tracing point advances by a corresponding movement from

P to B; the angular and progressive spaces are hence proportional to the time of description, and therefore proportional to each other.

As the radiant continually augments, so the angle about the pole acquires a similar increase. At every revolution, the radiant gains equal accessions, (fig. 209.) AB, BC, BD, &c. and each of these portions must be to the radiant as four right angles to the elapsed angle.

Cor. A right angle being denoted by 1, and any other elapsed angle, whether greater or less than a circuit, by φ, the corresponding radiant being expressed by R, while r denotes the radiant of a whole revolution; then will $R = r.\frac{\varphi}{4}$.

PROP. II. PROB.

To divide a given angle into any number of equal parts, by help of the Equable Spiral.

Let it be required to divide a given angle, suppose into three equal parts, by the application of this Spiral.

Draw the radiant PB (fig. 210.), making with the axis of the Spiral an angle APB, equal to the given angle, trisect PB in the points F and G, and from P as a centre describe the arcs FH and GI, and draw the radiants PH and PI, which will divide the angle APB into the equal angles APH, HPI and IPA.

B

For, by the last proposition, PA : PG : : APB : APH, and PF being the third part of PA, the angle APH must be one-third of APB. Again, PA : PG : : APB : API, and PG being two-thirds of PA, the angle API must likewise be two-third parts of APB.

PROP. III. THEOR.

The subtangent to any point of the Spiral is equal to the arc of a circle described by the radiant.

If the line BT (fig. 210.) touching the Spiral in the point B, be produced to meet the perpendicular PT, this subtangent will be equal to the elapsed arc AB, described by PB as a radius.

For draw the proximate radiant P*b*, and describe the minute arc Pβ. The elementary triangle Bβ*b* is evidently similar to the right-angled triangle PTB, and therefore β*b*, the increment of the radiant PB, is to βB, the increment of the elapsed arc ABP, as PB to PT, the subtangent; but, from the first proposition, β*b* is to βB as the radiant PB itself is to the elapsed arc AB, and consequently PB is to AB as PB to the subtangent PT, which must thus be equal to the elapsed arc AB.

Cor. Hence at the termination of the first, second, third and fourth, &c. revolutions, the corresponding subtangents are as the series of square numbers, 1, 4, 9, and 16, &c.; for the second, third, fourth, &c. subtangents are equal to two, three, four, &c. circumferences, whose radii are at the same time doubled, tripled, quadrupled, &c. In general, the radiant of a complete revolution being denoted by r, the angle elapsed in reference to a right angle by φ, and the circumference of a circle whose diameter is unit by π; then will the subtangent corresponding to one revolution be $2\pi r$, and the subtangent corresponding to any other polar angle will be $2\pi r.\left(\frac{\varphi}{4}\right)^2 = \frac{\pi r}{8}.\varphi^2$, or expressed in terms of its radiant R, the subtangent is $\frac{\pi\varphi}{2}$.R.

PROP. IV. THEOR.

If the Equable Spiral be divested of its radiating structure, it will become transformed into a parabola of the same corresponding dimensions.

Let there be always taken (fig. 211.) half of the subtangent PT for the absciss KM, and the radiant PB itself for the ordinate ML, and a parabola will arise which is equal in every respect to the Equable Spiral.

For draw the normal BE, and let PA be the direction of the axis. Call the radiant of the first revolution R, the circumference of a circle described by it C, the circumference of the circle described by the radiant PA being *c*. From the genesis of this Spiral, *c* is to the arc AB, or the subtangent PT, as R is to PA, and alternately *c* is to R as PT to PA; but the right-angled triangles TPB and BPE being similar, PT : PA : : PA : PE; and consequently *c* is to R as PA to PE, or alternately *c* is to PA as R to PE. But the circumference *c* is evidently to its radius PA as C to R, and therefore C : R : : R : PE; whence C and R being both given, the third term PE of the proportion is likewise a given line. Now, the right-angled triangle TBE being divided by a perpendicular, $PB^2 = PT.PE = \frac{1}{2}PT.2PE$; whence PB and the half of PT are the absciss and ordinate of a parabola which has

2PE for its parameter or PE for its semiparameter. In this parabola, MN and MO are equal to the tangent BT and normal PE, the subtangent NM is double of the absciss KM, and the subnormal MO, being equal to PE, is the semiparameter.

Cor. Hence conversely, if the absciss KM of the parabola be divided into minute spaces K*a*, *ab*, *bc*, &c. and ordinates erected from the points of section, and diagonals drawn; the sectors K*f*, *fag*, *gbh*, &c. being placed together about P, will form the representing spiral, which must therefore contain half the parabolic space KLM.

PROP. V. THEOR.

The length of a portion of the Equable Spiral is the same as that of an arc of the corresponding parabola.

Make the circumference of a circle (fig. 212.) to its radius, as the first radiant PA to PE or MO the semiparameter of the parabola, in which insert the ordinate ML equal to the radiant PB; then will the part of the spiral line, sweeping from its pole to the point B, be equal to the parabolic arc KL.

For assume the proximate radiant P*b*, and make the ordinate *ml* equal to it. The elementary triangles Bβb and Lλl, being similar to the equal triangles TPB and NLM, and having also the equal sides βb and λl, are therefore equal, and hence the element B*b* of the spiral is equal to the element L*l* of the parabolic arc. Wherefore the spiral branch PB, which results from the addition of B*b*, is equal to the curved line KL, which likewise proceeds from the corresponding accessions of L*l*.

PROP. VI. THEOR.

The space included by a radiant of the Equable Spiral is the third part of the area of the elapsed circular sector.

The space between spiral sweep and the radiant PB (fig. 212), is the third part of the area of the sector APB.

For the elementary triangles PB*b* of the spiral is evidently half of the elementary rectangle L*m* of the parabola, and therefore the spiral space is half of the parabolic segment KML, or equal to the exterior segment KQM, and is consequently the third part of the circumscribing rectangle KQML. But this rectangle is equivalent to the sector APB; for the side ML is equal to the altitude PA or PB, and KM is equal to the half of AB the circular base. Whence the spiral space PB is the third part of the area of the circular sector ABP.

Cor. Hence the whole spaces described by the radiants of the Equable Spiral at the end of the first, second, third, and fourth revolutions, &c. are as the series of squares 1, 4, 9, 16, &c., and therefore the intervening spaces are as the successive odd numbers 3, 5, 7, &c.

PROP. VII. PROB.

To find the circle of osculation at any point of an Equable Spiral.

Let it be required to describe (fig. 213.) a circle that shall osculate with the Spiral at the point B.

Let BT be the tangent, and BE its normal, and assume the proximate point *b* and the corresponding tangent *bt* and normal *be*. The subtangents being always equal to the elapsed arcs, the excess $t\tau$ of Pt above PT is equal to the excess of the proximate arc *ab* above AB. But this excess must be composed of $B\beta$ and of the excess of the arc $a\alpha$ above AB; make $\tau\theta$ equal to $B\beta$, and join $T\theta$. It is obvious that the radius PB is to the arc AB or the subtangent PT as βb to $\tau\theta$, the excess of $a\alpha$ above AB; but, the elementary triangle $b\beta B$ being similar to BPT, PB is to PT as βb to $B\beta$, which is hence equal to $\tau\theta$, the increments τt of the subtangent being thus double of $\tau\theta$ or $B\beta$.

The right angled triangle BPT is evidently similar to $Tt\theta$, and therefore $BT : PT :: t\theta : T\tau$, or alternately $BT : \tau\theta :: PT : T\tau$; whence the minute angle $TB\theta$ is equal to TPt or BPb. But this angle is likewise equal to BEb, for the triangles $B\beta b$ and BPE being similar, $BP : BE :: B\beta : Bb$, and alternately $BP : B\beta :: BE : Bb$; wherefore if $TB\theta$ had been the whole increment which the tangential angle received, the normals from B and *b* would have then met in E, and assigned this point as the centre of the osculating circle. But the angle made by the tangent receiving a larger increment TBt, the normals BE and *be* acquire an equal inclination Bsb, and therefore they unite sooner and indicate a greater curvature.

Produce TP till EQ be equal to PE, and upon BE let fall the perpendicular QR. It is obvious that the triangles PBE and BEQ are similar to $T\tau\theta$ and $T\theta t$; whence BE is to BR as $T\theta$ to $T\varepsilon$, and as the angle $TB\theta$ or BEb to $TB\varepsilon$ or Bsb; but the angle BEb is to Bsb as BS to BE, and therefore BE : BR : : BS : BE, or alternately BR : BE : : BE : BS.

Hence the centre of curvature is readily found; for, perpendicular to the radiant, draw PE to meet the normal, and produce it to an equal distance EQ, let fall the perpendicular QR, and make BS a third proportional to BR and BE; the point S is the centre of the osculating circle.

Cor. Hence the radius of curvature at any point B of this Spiral, is very nearly equal to a perpendicular BV let fall upon the axis.

The Equable Spiral was first proposed by Conon of Samos to his friend Archimedes, who displayed the resources of his mighty genius in the spring of Geometrical Science by disclosing the chief properties of the curve. It is hence commonly named the *Archimedean Spiral.* The very curious property of its assimilation to the Parabola was discovered, before the middle of the seventeenth century, by Gregory of St Vincent.

XVI. HYPERBOLIC SPIRAL.

If, from a point in a straight given by position, arcs be described of a given length, the locus *of their extremities is the* R*ECIPROCAL* *or* H*YPERBOLIC* S*PIRAL.*

Let AB, CD, EF, &c. (fig. 214.) be arcs all of the same length described from the point P, their summits A, E, &c. will mark out the *Reciprocal or Hyperbolic Spiral.*

DEFINITIONS.

1. The centre P of the successive arcs is called the *Pole* of the curve.
2. The straight line PF on which those arcs are constructed is called the *Axis.*
3. A straight line PE drawn to any point B of the curve is a *Radiant*, a perpendicular EI to the axis *an Ordinate*, the intercepted portion PI being the corresponding abscise.
4. By an extension of analogy, a perpendicular PT erected from the pole to meet a tangent AT, may be called a *Subtangent.*

PROP. I. THEOR.

The radiants of the Hyperbolic Spiral are inversely as the angles which they make with the axis.

Let (fig. 214.) PA and PC be any two radiants of the curve; then PA will be to PC reciprocally as the angles BPA and DPC, or as the angle DPC is to the angle CPA.

For the concentric sectors BPI and DPC being similar, PI or PA is to PC as the arc BI is to the arc DC, which, from the nature of the curve, is equal to BA; consequently PI is to PC as BI to BA, that is, as the angles BPI or DPC is to the angle BPA.

Cor. Hence a given angle may be divided into any number of equal parts. Thus, let it be required to trisect the angle APB. Take PF=3PB, describe the arc FE; again, make PB : PD : : 2 : 3, describe the arc DC, and join PC. The three angles APC, CPE and EPF are evidently all equal.

Scholium. Hence the radiants and the angles which they make with the axis, stand in the same relation to the *abscissæ*, and the ordinates of the hyperbola applied to the asymptote. The name of *Reciprocal* or *Hyperbolic Spiral* is therefore sufficiently appropriate. As the radiants expand, they become always less inclined to the axis of the curve. This spiral must hence bend continually to a parallelism with the axis, from which its distance can never exceed the length of an arc AB. If, therefore, a perpendicular DG be made equal to AB, the parallel GH will be an asymptote to the curve.

PROP. II. THEOR.

A subtangent to the Hyperbolic Spiral, is equal to the length of the constituent arc.

Let AT (fig. 214.) touch the curve at A and meet PT a perpendicular to the radiant PA; the segment PT will be equal to the length of any intercepted arc AB.

For assume the proximate radiant Pa, and describe the arc $a\alpha$C. The right-angled triangle APT is obviously similar to the elementary triangle Aαa, and consequently PA : PT : : $a\alpha$: αA, or alternately PA : αa : : PT : αA. But αPB and aPb being concentric sectors, Pα or PA is to Pa as the arc αB to ab or AB, and by division PA is to $a\alpha$ as AB to αA. Whence, by identity of ratios, PT is to αA as AB to the same αA, and therefore PT is equal to the arc AB.

Cor. Hence the tangent of the deflexion of the curve from a radiant is proportional to the angle which the radiant itself makes with the axis. For AP being the radius, PT is the tangent of the deflexion PAT, while the arc AB, which is equal to it, measures the angle BPA.

PROP. III. THEOR.

Any sector of the Hyperbolic Spiral is equivalent to half the rectangle under a radiant and the length of the constituent arc.

The space inclosed from the origin of the curve to the radiant PA (fig. 214.) is equivalent to half of the rectangle under this radiant, and AB or PG.

For draw the proximate radiant Pα, and from the pole describe the minute arc *ab*, and erect PT perpendicular to the tangent. The elementary triangle AP*a* is equivalent to half the rectangle under its base PA and the altitude αA; but the right-angled triangle Aαa being evidently similar to PAT, PA : PT : : $a\alpha$: αA, and PT.$a\alpha$ = PA.αA. Wherefore the elementary triangle AP*a* is equivalent to half the rectangle under the constant subtangent PT and Aα the increment of the radiant. The collective amount of AP*a*, or the spiral space limited by the radiant PA, is hence equivalent to half the rectangle under PT and the whole extent of the spiral from its origin, or equivalent to the right-angled triangle APT.

Cor. Hence the spiral space included between the radiant PA and PC, and between this and the radiant PE, are equivalent to the rectangles under PT, and the excess of PC above PA, and of PE above PC.

PROP. IV. THEOR.

If the Hyperbolic Spiral were divested of its radiating structure, it would be converted into an equal Logarithmic Curve, having the same subtangent, but the radiants changed into ordinates.

If the radiants PA (fig. 214.) were placed perpendicularly to the base CA (fig. 170.) and at mutual intervals equal to αa, their summits would range in a Logarithmic Curve representing the Hyperbolic Spiral.

For the elementary triangles Aαa and $m\mu$M would evidently be equal, and likewise the right-angled triangles APT and MBR; consequently the subtangent BR would be equal to PT and therefore constant, which is a distinguishing property of the Logarithmic Curve.

The length of this curve is hence equal to that of the Hyperbolic Spiral, and the interval between two ordinates is equal to the arc intercepted between the corresponding radiants. The area of the Logarithmic Curve, however, is double of the corresponding space in the spiral; for, being equivalent to the rectangle under BM and BR, it is double of the triangle MBR or APT, the measure of the space described by the radiant PA.

The Hyperbolic Spiral appears to be the invention of James Bernoulli. Its properties are not numerous, but they serve to mark the contrast and analogy of a class of related curves. The Logarithmic Spiral had been demonstrated in the *Principia* to be the Trajectory which corresponds to an attractive force inversely as the cube of the distance. James Bernoulli discovered that the Hyperbolic Spiral would have answered the same law. Such is the confusion which would have resulted in the System, whether the planets were scattered by spreading convolutions, or were carried away nearly in determinate directions through boundless space.

XVII. LOGARITHMIC SPIRAL.

If an extended straight line turn uniformly about its extremity, a point which is carried along it, with a celerity proportioned to the distance from that centre, will describe the LOGISTIC *or* LOGARITHMIC SPIRAL.

Let the indefinite line (fig. 215.) PALY revolve uniformly about a fixed point P, while another point A either advances or recedes on that line, but with a celerity always as its distance from the common centre of motion; it will trace out the *Logistic* or *Logarithmic Spiral.*

When the point A advances along the revolving line, it is obvious, that the spiral must swell continually outwards; but when that tracing point recedes, the curve will perpetually contract in a constant approximation to the centre.

DEFINITIONS.

1. The centre of revolution is called the *Pole* of the Logarithmic Spiral.
2. Any straight line drawn from the pole to a point in the curve is called *a Radiant.*
3. The complement of the angle which a tangent makes with the corresponding radiant is termed *the Angle of Deflexion.*

PROP. I. THEOR.

Radiants which form equal angles, about the pole of the Logarithmic Spiral, are continued proportionals.

Let (fig. 215.) the polar angles ABP, BPC, CPD, &c. be all equal; then will PA : PB : : PB : PC : : PC : PD, &c. For make Px=PA, Py=PB, Pz=PC, &c. Since the angles of revolution are equal, the linear spaces xB, yc, zd, &c., described in the same time, by the motion of the point along the radiant, must be proportional to the distances Px, Py, Pz, from the pole; or Px : xB : :Py : yC : : Pz : zD. Whence by composition Px or PA : PB : : Py or PB : PC : : Pz or PC : PD, &c.

Cor. 1. It is evident, that if equal angles APb, bPc, cPd, &c. be retraced, the radiants PA, Pb, Pc, Pd, &c. will form a descending progression which is likewise in continued proportion.

Cor. 2. The angles about the pole are thus the logarithms of the ratios of the successive radiants, and hence the name bestowed on this curve.

PROP. II. THEOR.

The tangent at any point of the Logarithmic Spiral, makes a given angle with the corresponding radiant.

Let the straight line AT (fig. 216.) touch this spiral at the point A; then will the angle PAT have invariably the same magnitude.

For assuming another radiant PB, draw the proximate radiants Pa and Pb, making equal elementary angles APa and BPb, and describe the minute arcs $A\alpha$ and $B\beta$.

Since the angles APa and BPb are equal, the radiants PA and PB must have received increments αa and βb proportional to themselves; but the elementary sectors being similar, PA is to PB as the minute arc $A\alpha$ to $B\beta$. Wherefore $A\alpha : B\beta :: \alpha a : \beta b$, and alternately $A\alpha : \alpha a :: B\beta : \beta b$; consequently the right-angled triangles $A\alpha a$ and $B\beta b$ are similar, and the angle $Aa\alpha$ is equal to $Bb\beta$. Whence the tangents at A and B, which indicate the flexures of the Spiral, make equal angles with their corresponding radiants PA and PB.

Cor. Hence the angle APB exhibiting the logarithm of the ratio of PA to PB, if PA represent unit, the arc AC described from P will express the logarithm of the radiant PB.

Scholium. It is obvious that the greater is the obliquity of this spiral, the more rapidly will its radiants acquire extension. While the logarithm of PA receives the increment $A\alpha$, the radiant itself gets the accession αa; but αa is to to αA as radius to the tangent of the angle $\alpha a A$ or PAT, which the curve makes with the radiant, and consequently the tangent of this angle indicates the *modulus* of the system of logarithms which belongs to the particular spiral.

The scale of these logarithms depends thus on the magnitude of the tangential angle. When that angle is half a right angle, as in fig. 216. the subtangent PT being now equal to the radiant PA, the spiral will exhibit the *Natural Logarithms.* In general, if PA denote the unit, the corresponding subtangent PT must express the *modulus* of the System of Logarithms.

Let t denote the tangent of the angle PAT, r and R any two radiants PA and PB, a the measure in degrees of their included angle APB, and π the circumference of a circle whose diameter is 1: then the natural logarithm of $\frac{R}{r} = \pi t . \frac{a}{180}$. Suppose the curve to cut its radiants at half right angles, and the natural logarithms of $\frac{R}{r}$, at the end of a half or of a whole revolution will be 3.1415926 and 6.2831853, the ordinary logarithms being 1.3643761 and 2.7287527. Whence in this species of the curve, the radiants would be multiplied in a half revolution 23,1472 times, and in a whole arc 535,4916 times; and it is easy to compute that they would be doubled at every polar angle of 39° 42′ 46″.4. In like manner, it might be shown, that the radiants are doubled only at the end of an entire circuit, when the angle of deflexion, or the complement of the obliquity of the spiral, is 6° 17′ 41″, but doubled at each revolution when that angle is 12° 26′ 30″.

PROP. III. THEOR.

The space limited by a radiant of the Logarithmic Spiral is equivalent to half the right-angled triangle formed by that radiant, the perpendicular to it drawn through the pole, and the corresponding tangent.

If PT (fig. 211.) be drawn at right angles to PA, and meeting the tangent AT; the sectoral space from the origin of the spiral to the radiant PA will be half of the triangle APT.

Let P*a* be a proximate radiant, and *aαp* parallel to AP; when the sectoral space becomes augmented by the elementary triangle AP*a*, the right-angled triangle APT erected upon the radiant PA passes into a similar triangle erected upon P*a* or the approximating perpendicular *ap*. The triangle APT therefore acquires the increment PA*ap* or AP*αp*, of which the increment AP*a* of the sectoral space is evidently the one half.

Cor. 1. Hence PA is to PT as $\frac{1}{4}$PA^2 to the space from the origin of the spiral to the radiant PA, and therefore 4AP is to PT as PB^2—PA^2 to the space included between any two radiants PA and PB.

Cor. 2. If the spiral traverse its radiants at half right angles, (fig. 216.) the triangle APT, having equal sides PT and AT, will contain half the area of the square of PA, and consequently the sectoral space will be equivalent to a square described upon the half of the radiant PA.

PROP. IV. THEOR.

The length of the Logarithmic Spiral from its origin is equal to the tangent, limited by a perpendicular to the terminating radiant through the pole.

The curve winding from the pole to A, (fig. 211.) is equal to the tangent AT, as determined by PT drawn at right angles to PA.

For the elementary triangle $A\alpha a'$ being similar to TPA, Aa is to αa as PT to PA; but PA is composed of the increments αa, and consequently PT must be the accumulation of Aa, which generates the extension of the curve. The spiral convolution from the pole to A is therefore equal to the tangent PT.

Cor. Hence the sectoral space is equivalent to the rectangle under the length of the spiral arc, and the fourth of the perpendicular PD let fall from the pole upon the tangent AT. For this sectoral space, being half of the area of the triangle APT, is therefore equivalent to the rectangle under the perpendicular PD and the fourth part of the base AT.

PROP V. THEOR.

The radius of curvature at any point of the Logarithmic Spiral is the normal.

Let AN (fig. 211.) be drawn at right angles to the tangent AT, and meeting the extension of a perpendicular PT to the radiant PA in N; this point N is the centre of the circle osculating at A.

For assume the proximate point a, and join aN. The right-angled triangle APN is evidently similar to $A\alpha a$, and consequently PA : NA : : $A\alpha$: Aa, or alternately PA : $A\alpha$: : NA : Aa. Wherefore the elementary sectors $AP\alpha$ and ANa are similar, and consequently the angle $AP\alpha$ or APa is equal to ANa. But since the tangential angle remains constant, the normals from A and a must have the same inclination as the radiants PA and Pa; whence aN is a normal, and the point N, where those perpendiculars AN and aN meet, is the centre of the circle which osculates at A.

PROP. VI. THEOR.

The Involute of the Logarithmic Spiral is the same curve.

If a thread lapped about the Logarithmic Spiral from P to A (fig. 217, and 218.) be unfolded, it will produce the very same curve.

For the developed line AT is evidently the radius of the circle osculating the involute at T; and consequently its tangent TV is a perpendicular to AT. Wherefore the angle PTV is the complement of PTA, and is hence equal to the given tangential angle PAT. But such is the characteristic property of the Logarithmic Spiral, and consequently this involute, expanding with the same angle, is only a copy of the original curve, though extended farther in the ratio of its radiant PT to PA.

Scholium. The Logarithmic Spiral may be described mechanically, by help of an instrument founded on the principle of the equality of the angles which this curve makes with its radiants. AB (fig. 208.) is a straight bar of brass, three quarters of an inch broad, one-eighth thick, and perhaps a foot in length. It is divided longitudinally by a groove, and terminated by a circular piece CDE, within which, a smaller circle of bell-metal is lodged, having a steel wheel with a sharp edge and about one quarter of an inch in diameter, implanted in the centre at right angles to its plane. The quadrant CD is marked with fine divisions, and the interior circle being turned into any given oblique direction is then secured by the pressure of two opposite screws. A strong pin P being stuck into the drawing board, and the groove of the ruler passed over it, the head A is made to advance under a gentle pressure, the wheel shaping its progress while BE glides close along P.

This instrument would evidently be of considerable use to Architects, for describing *Scrolls* and *Volutes*, which in ordinary practice are not drawn after any correct principle. The circle CDE is the same with that which commonly occupies the origin of the Volute.

But a very near approximation to the Logarithmic Spiral may be derived from the remarkable property,—that the normal is always equal to the radius of curvature—by connecting the similar minute arcs of successive circles. Let a circuit (fig. 219.) be divided into any number of equal angles APB, BPC, CPD, &c. about the point P. Beginning from the point A, draw A*a* parallel to PF next the opposite radiant PG, to meet the perpendicular PD, and from *a* with the radius *a*A describe the arc AB; now draw B*b* parallel to PG next its radiant PH to meet the perpendicular PE in *b*, from which point with the radius *b*B describe the arc BC. Continue this process by describing arcs from the successive centres *c*, *d*, *e*, &c. It is evident that A*a*, B*b*, C*c*, D*d*, &c. are normals or perpendiculars to the tangents at A, B, C, D, &c. These points are hence accurately assigned, and the small aberrations of the Spiral arising from the sudden changes of curvature could easily be corrected by a slight touch of the pencil at the various inflexions.

It is manifest that the centres *a*, *b*, *c*, *d*, &c. of the osculating circle will mark out the Evolute of the curve, which is the very same Spiral, though differently placed and less developed.

The figure thus produced by a succession of coalescent arcs described from a series of interior centres, exactly resembles the general form and the elegant *septa* of the *Nautilus*, which might be adopted with great effect in many architectural ornaments.

The Logarithmic Spiral was first incidentally noticed by the illustrious Des Cartes, who, with his usual acuteness and rapidity, discovered its principal features,—the equality of the angles which the curve makes with its radiants, and the constant ratio of its whole preceding length to any radiant. The profound sagacity of Newton arrived at the remarkable conclusion, that had the force of gravitation been inversely as the *cube*, instead of the *square* of the distance, the planets would have all shot off from the sun in diffusive Logarithmic Spirals, the eye of science thus catching a glimpse of that High Design which presides in the actual constitution of the Universe. James Bernoulli investigated the more abstruse properties of this Spiral, and was delighted with contemplating its *perpetual renascence*, or its faculty of reproducing constantly the same curve, though transformed by evolution, or changed into a caustic, either of reflexion or of refraction. Warmed with the enthusiasm of genius, he desired, in imitation of Archimedes, to have the Logarithmic Spiral engraved on his tomb, and directed, in allusion to the sublime tenet of the Resurrection of the Body, this emphatic inscription to be affixed—EADEM MUTATA RESURGO.

NOTES.

Note 1.—Page 184.

The three books, of which this first treatise consists, are designed to exhibit a distinct and comprehensive view of the mode by which the Greek geometers conducted their Analysis. For that purpose, I have chosen a series of propositions, at first extremely simple, but gradually rising in difficulty as the train of investigation proceeds. The first book, being rather of a miscellaneous nature, is drawn from a variety of sources, and now contains a copious collection of the most interesting problems. The 31st and 32d Propositions exhibit the different analyses of the two problems so famous in the Platonic school —*the trisection of an angle*—and *the duplication of the cube*— which led immediately to the cultivation of the higher geometry.

In the second and third books, I have endeavoured to comprise all that relates to the ancient analysis in its most improved state, as extended by the labours of Apollonius and his illustrious contemporaries. Without omitting any material proposition, I have yet avoided the prolixity of pursuing in detail their numerous subdivisions. Our system of modern education, embracing such a wide range, would scarcely indeed afford leisure for indulging in those easy tasks.

The method of analysis, so deservedly valued in the ancient schools, was regularly studied after the Elements of Geometry. According to Pappus, it consisted of eight distinct treatises:

1. The *Data*—περὶ τῶν δεδομένων—in a single book of consi-

derable length, but containing propositions only of the very simplest kind.

2. The *Section of Ratio*—περὶ λογȣ ἀποτομῆς—in two books, which Dr Halley, with much sagacity and incredible labour, restored, from a MS. in the Bodleian library. The object of the tract was the solution of this problem, branched out into a multitude of cases, and marked with various limitations: "Through a given point to draw a straight line intercepting segments on two straight lines which are given in position, from given points and in a given ratio." It forms the first four propositions of the second book.

3. The *Section of Space*—περὶ χωριȣ ἀποτομῆς—in two books. Of these no vestige remained; but Dr Halley, guided by a few hints from Pappus, very successfully exerted his ingenuity in divining the original structure. It was proposed—"Through a given point, to draw a straight line cutting off segments from given points on two straight lines given in position, and which shall contain a rectangle equal to a given space." This occupies the propositions from the 5th to the 10th inclusively of the second book.

4. The *Determinate Section*—περὶ διωρισμένης τομῆς—in two books. These were also lost; but Dr Simson, assisted by the attempts of Schooten, has restored them in the most luminous manner. Their object was—"To find a point, the rectangles or squares of whose distances from given points in the same straight line should have a determined ratio. They form Prop. 10.—18. Book II.

5. *Inclinations*—περι νεύσεων—in two books. It was proposed—"To insert a straight line, of a given magnitude, and tending to a given point, between two lines which are given in position." This problem was restored by Marinus Ghetaldus, a patrician of Ragusa; and other investigations were given by Hugo d'Omerique, in his ingenious treatise on Geometrical Analysis, printed at Cadiz in 1698. Two solutions of the case of the rhombus, remarkable for their elegance, appeared in the posthumous works of Huygens, who was imbued with the finest taste for the ancient geometry. I have condensed the whole in Prop. 24—30. Book II.

6. *Tangencies*—περὶ ἐπαφῶν——in two books. Of this tract only some lemmas were preserved, which enabled the celebrated Vieta in a great measure to restore it. Some of the cases which had escaped him were solved by Marinus Ghetaldus; and farther improvements were made in 1612, by Alexander Anderson of Aberdeen, an ancestor of the Gregorys. The general problem occupies the remainder of the second book.

7. *Plane Loci*—περὶ τόπων ἐπιπέδων—in two books. The object was—"To find the conditions under which a point, varying in its position, is yet confined to trace a straight line or a circle given in position." This beautiful train of investigation was partly restored by Schooten in 1650, though after a sort of algebraical form. The ingenious Fermat succeeded in bestowing greater simplicity on the subject. But all these attempts have been eclipsed by Dr Simson, whose treatise *De Locis Planis*, published at Glasgow in 1749, is a model of geometrical strictness and elegance. The first 21 propositions of the third Book include all the principal theorems, which I have selected with additions.

The six preceding branches of analysis were all the creation of Apollonius of Perga, the most assiduous and inventive of the Greek geometers.

8. *Porisms*—περὶ τῶν πορισμάτων—in three books, composed by Euclid. No trace of these now remains, except some obscure hints of Pappus, rendered still more perplexed by the corrupt and mutilated state of his text. The subject had long proved an ænigma which it baffled the efforts of the ablest and most learned mathematicians to unravel. Fermat advanced some steps; but the honour of completing the discovery was reserved for our countryman Dr Simson, whose restoration of the Porisms was given to the scientific world in 1776, in a posthumous volume, printed at the expence of the late Earl Stanhope. From that work I have extracted what seemed the best suited to my purpose; and I have likewise availed myself of the judicious remarks and illustrations of my late distinguished and most lamented colleague, Professor Playfair. These porisms, with some additions, are contained in Prop. 22–32. Book III.

The remaining propositions of the third book relate to the subject *Isoperimeters*; which I have treated with the conciseness of the moderns, without departing, I hope, from the rigour of the ancient geometry.

Note 2.—Page 43.

It is obvious, that Prop. 28. is the same, as to find the intersection of a given ellipse with a straight line given by position. The analysis of this problem, it may be observed, brings out the fundamental property of Lines of the Second Order; as the composition furnishes another demonstration of the equality of the transverse axis to the sum or difference of radiants from the *foci*, which may be extended analogically to the parabola.

Note 3.—Page 45.

This proposition is only a very limited case of the general problem of *inclinations*, which occupies inclusively from the 24th to the 30th Propositions of the Second Book of Analysis. The construction given in the text is immediately deduced from the second solution of Prop. 30. Book II.

Note 4.—Page 49.

This and the next problem, which has been ascribed to a response of the *Delian* oracle, called forth the powers of the ancient analysis, and transcending the limits of elementary construction, led to the discovery of some of the higher curves, and essentially contributed to the extension of geometrical science. For the trisection of an angle, Nicomedes proposed the *conchoid*, the properties of which are explained in the Treatise of the Higher Curves. An elegant solution of the problem is given in Newton's *Universal Arithmetic*, by means of an hyperbola whose asymptotes form an angle of 120°. It constitutes Prop. 25. Book II. of the Geometry of Lines of the Second Order.

Note 5.—Page 51.

In this proposition, I have condensed and endeavoured to simplify the fine speculations of the Greeks, respecting the duplication of the cube. The first analysis is that given by Hero, in his *Mechanical Institutions;* and the variation of it was proposed by Philo of Byzantium. The second analysis of the problem was given by Nicomedes, and the third by Pappus of Alexandria. In the first and second methods, the solution may be performed by the conchoid; in the third, it is effected by the *cissoid* of Diocles, as demonstrated in the Treatise of the Higher Curves. Menechmus solved the problem in two ways—either by combining two parabolas—or by combining a parabola with a rectangular hyperbola. It forms, Prop. 24. Book II. Geometry of Lines of the Second Order.

Note 6.—Page 85.

This theorem is the only one supplied by the *Data* of Euclid. Since the angle BDF is half of the angle ABC, and $\mathrm{DB} : \mathrm{BF} :: R : \mathit{tan}\,\mathrm{BDF}$, it follows that, $4R : \mathit{tan}\,\frac{1}{2}\mathrm{ABC} :: (\mathrm{AB}+\mathrm{BC})^2-\mathrm{AC}^2$: area of the triangle, or, by decomposition, $R : \mathit{tan}\,\frac{1}{2}\mathrm{ABC} :: \left(\frac{\mathrm{AB}+\mathrm{BC}+\mathrm{AC}}{2}\right)\left(\frac{\mathrm{AB}+\mathrm{BC}-\mathrm{AC}}{2}\right)$: area of the triangle. It hence follows that, assuming the former notation, $\mathrm{T}=s(s-\mathrm{AC})\,\mathit{tan}\,\frac{1}{2}\mathrm{ABC}$. This property is adopted in the last edition of the Elements of Plane Trigonometry.

Note 7.—Page 101.

This and the six preceding propositions include those cases of the problem of *inclinations* which admit of an elementary construction. The first solution is borrowed from the geometrical analysis of Hugo d'Omerique, and the second from the posthumous works of the celebrated Huygens. To solve the general problem would require the application of the conchoid.

Note 8.—Page 135.

This proposition, extended to points in different planes, furnishes a legitimate demonstration of the remarkable property of projected masses, which forms, in Newton's *Principia*, the fourth corollary to the laws of motion; namely, that of any system of bodies impressed with uniform and rectilineal motions, the centre of gravity either remains at rest or travels uniformly in a straight line.

Note 9.—Page 153.

It is easily perceived from the mode of successive construction, that the centre of the circle which terminates this process must likewise be the centre of gravity of the several points. This curious property is noticed in Huygens' elegant tract, entitled *Horologium Oscillatorium*, and furnishes another example of the application of the principle of the *conservatio virium vivarum*, which has such extensive influence in the mutual action of bodies.

Note 10.—Page 156.

The porismatic point D is the centre of gravity of particles of matter situate at A, B and C; for MN being any straight line drawn through D, the distance CG is equal to the combined distances AH and BI, and consequently the opposite efforts of those particles, to turn their plane, must, about the centre D, maintain every way an exact equipoise.

The proposition might easily be extended to any number of points in the same plane; but it is true universally, if the points only have a determined position. The writers on Statics, however, have commonly assumed, what they were not entitled to do, the existence of an individual centre of gravity. This fundamental property of matter is simply and elegantly demonstrated by the ingenious Boscovich, in his *Theoria Philosophiæ Naturalis*, a work of very great and original merit.

Note 11.—Page 173.

This problem was first proposed by Sir Isaac Newton, for determining the path of a comet, from four observations made at given short intervals of time. But unfortunately it was afterwards found in practice to give uncertain or even erroneous results. This unexpected failure led Boscovich to examine closely the circumstances which might affect the solution, and he discovered that the problem becomes indeterminate or porismatic, in the very case where its aid is wanted to guide astronomical observation.

Note 12.—Page 247.

It might likewise be shown, in addition to these corollaries, that the square of the semiconjugate axis OB (fig. 53.) is equivalent to the rectangle under the normal CR, and the perpendicular ON let fall from the centre to the tangent. For the triangle PCR being evidently similar to ONS, it follows that CP or OQ : CR : : ON : OS, and therefore OQ : SO = CR : ON; but OB is a mean proportional to OQ, and OS or $OB^2 = OQ.OS$, and hence $OB^2 = CR.ON$.

Having drawn HL (fig. 65.) from the centre of the osculating circle perpendicular to the focal chord CF, the right-angled triangles HLC and FPC are evidently similar, and CL : CH : : PF : CF : : OB : OD. But AO.OB=ON.OD, and hence OB : OD : : ON or CK : AO, and CK : AO : : CL : CH; wherefore $CH.CK = AO.CL = OD^2$.

Again, because $OB^2 = CR.CK$, it follows that $OB^2 : OD^2$: : CR.CK : CH.CK : : CR : CH; but it was proved that OB : OD : : CL : CH, or $OB^2 : OD^2 : : CL^2 : CH^2$, and therefore CR : CL : : CL : CH, whence the triangles CRL and CLH are similar, and RL perpendicular to CF. From this property the centre of the osculating circle is found by an easy construction. For draw CR at right angles to the tangent, let fall the perpendicular LR on the focal chord, and erect HL perpendicular to it; the point of intersection H will be the centre required.

The same process will evidently extend to the parabola; for the normal DM (fig. 66.) being drawn perpendicular, and RL parallel to the tangent, the perpendicular LH to the focal chord will assign the point H, the centre of osculation. This inference might also be drawn immediately from the nature of the curve, since DM bisects the angle FDX, the triangles DLH and DXH are equal, and so are DML and MDT; whence $DX = DL = TM = 2\,TF$ or $2DF$, the semiparameter of the diameter DX.

The properties now explained afford a simple investigation of the laws of centripetal forces. 1. Let a point C (fig. 65.) describe an ellipse under the controul of a force directed to the centre O. This centripetal force will then be inversely as the ratios compounded of the square of the perpendicular CK upon the tangent, and of CG half the intercepted chord of the equicurve circle. But the rectangle CK,OD being equivalent to AO,OB a given space, CK is hence inversely as OD; wherefore the centripetal force is directly as the square of OD, and inversely as CG. But $CO : OD :: OD : CG$, and $CO^2 = DO.CG$; whence the centripetal force being directly as the rectangle CO,GC, and inversely as the side CG, must be directly as the other side CO, the distance from the centre of tendency. This result applies to the case of rotatory pendulums describing very small curves, which must be elliptical if considered as in the same plane.

2. Let the point C be carried in an ellipse by a force emanating from the focus. The centripetal force will now be in the inverse ratio compounded of the square of the perpendicular FP and CL, half the intercepted chord of the equicurve circle. But it was proved, that $CL.AO = OD^2$; whence AO remaining constant, CL must be proportional to OD^2. Wherefore the centripetal force must be inversely as the squares of FP and OD, or in inverse duplicate ratio of the rectangle FP,OD. Now $FP : CF :: OB : OD$, and $FP.OD = FC.OB$; or, OB being invariable, the rectangle FB.OD is proportional to FC. Consequently the centripetal force is inversely as the square of FC, the distance from the centre of attraction; being the great law which not only guides the revolutions of the

planets about the sun, but the revolutions of their satellites about themselves.

Note 13.—Page 297.

The problem, to find the intersection of an oblique line AP with a given parabola, without describing the curve, was first incidentally solved by Torricelli, the famous inventor of the barometer. His object was to discover the point where a ball projected from a cannon with a given charge and in a given direction, and therefore describing a parabola of which the directrix and a tangent are given, would strike a plane having a given inclination. The construction proposed by that elegant geometer, is the same as the second one demonstrated in the text: For AL being the line of projection, AI the horizontal range, and AP the oblique plane; IN is drawn perpendicular to the directrix, NM parallel to the tangent, and MP, drawn again perpendicular to the tangent, assigns the point P.

The same conclusion is derived from the elementary property of the parabola. Since AL : AO : : LI OM or LN, and AL : AO : ; LN : OP; by composition AL^2 : AO^2 : ; LI ; OP.

Note 14.—Page 267—299.

Prop. 5. to 22. Book II. contains the very curious and interesting theory of the organic description of curves, on which the genius of Newton flashed a sudden blaze. The subject was afterwards pursued in its details by Maclaurin and by Braickenridge, but with very little regard to clearness or geometrical taste. The beautiful property,—that if the opposite sides of a hexagon inscribed in a Line of the Second Order be produced, the points of concourse will range in a straight line,—which flows from the same principles, was discovered by the penetration of Pascal, who made it the ground of a system of those curves, composed in early youth.

The description of Lines of the Second Order by the intersection of radiants turning about given poles, has lately been revived and extended by the celebrated Carnot and some other French Mathematicians, under the name of *Transversals.* This mode of developing the curves, not only sets them in a new light, but greatly facilitates the solution of various problems, and may therefore improve the practice of architecture and perspective.

Note 15.—Page 303.

This corollary elucidates the Theory of Equations. It is well known that the intersections of a circle with a parabola determine the roots of a biquadratic equation which wants the second term. But the additive and subtractive roots must in this case produce a mutual balance. Accordingly, the ordinates CR and DS are together equal to the opposite ordinates AP and BQ.

Note 16.—Page 331–336.

The *Conchoid* is easily described mechanically, by means of an instrument called the *Trammel of Nicomedes.* This consists of a long ruler CDC′ (fig. 162, 163, 164.) divided longitudinally by a groove, with a perpendicular piece DP, having a steel pin P, which can be screwed at any place. Another ruler BAP has a pencil that slides into any situation B, a steel pin fixed constantly at A, and then a groove stretching beyond P. The pin P and the pencil B being secured in their proper places, the pin A is inserted in the groove CD, while the groove AP is passed over the pin P, and the pencil B now traces the Conchoid, which is exterior or interior according as B is taken above or below the directrix CDC′.

The portion of an exterior Conchoid from the vertex to near its point of contrary flexure, is assumed by some architects for the outline of the shaft of the Corinthian column, the directrix representing the axis. A large temporary Trammel is therefore in practice employed to trace the segment of the curve.

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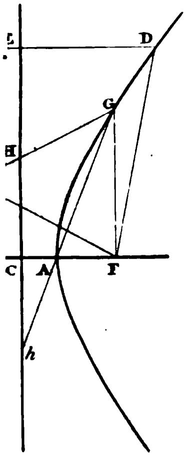
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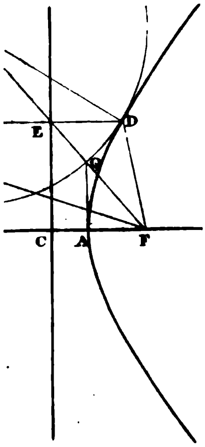
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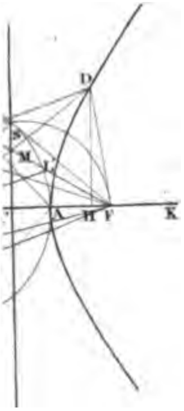
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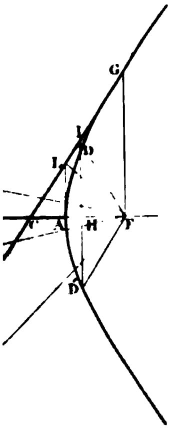
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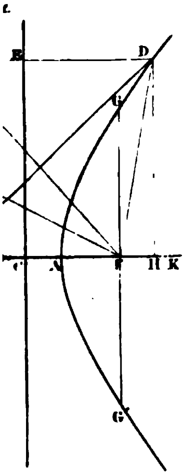
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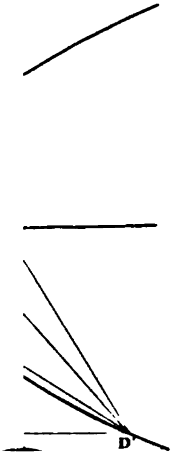
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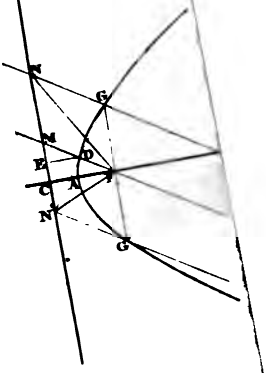

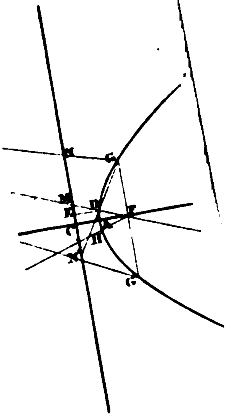

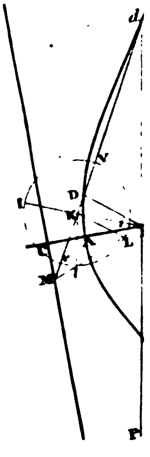

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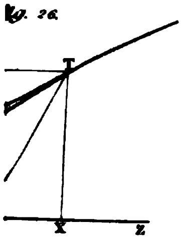
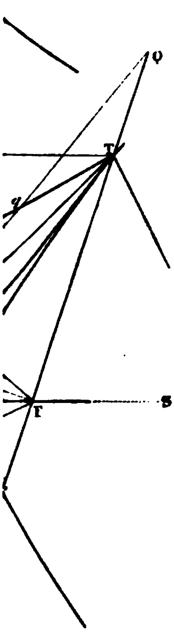
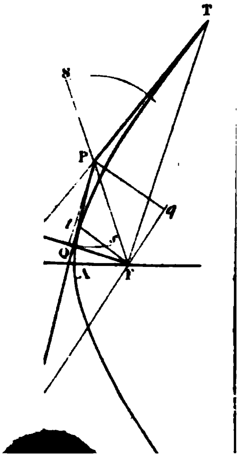
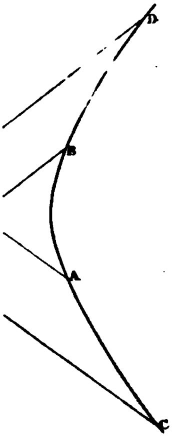
Fig. 26.









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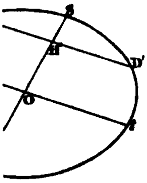

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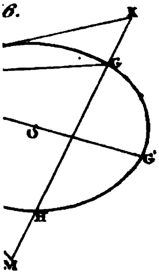

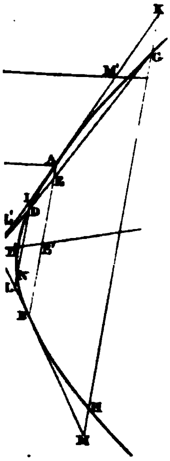

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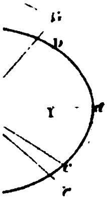

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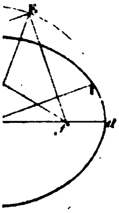

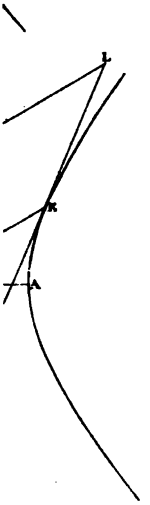

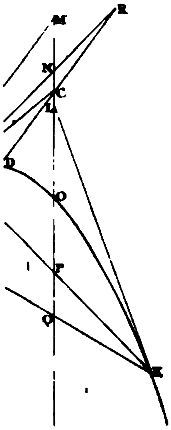

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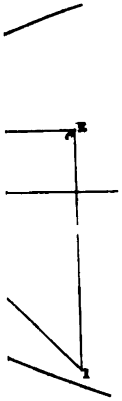

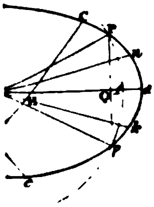

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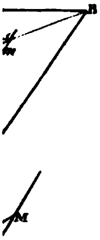

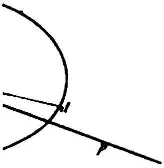

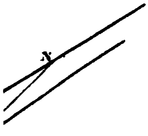

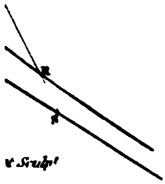

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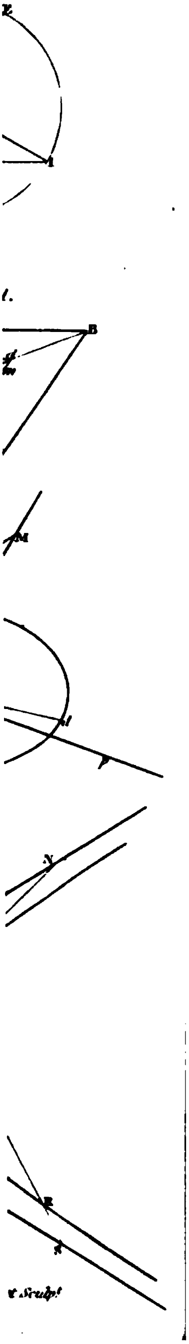
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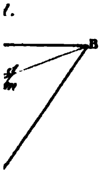

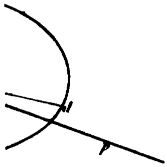

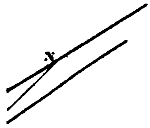

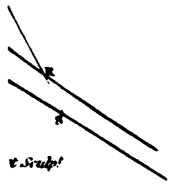

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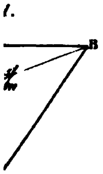

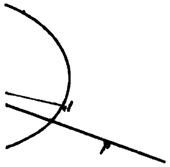

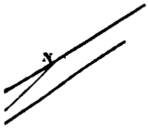

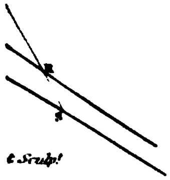

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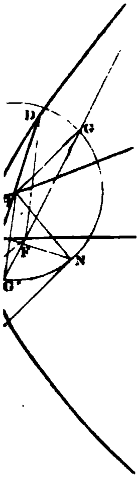
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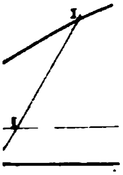
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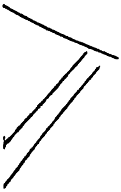

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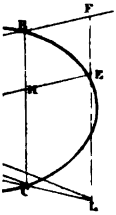

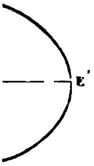

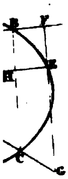

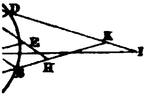

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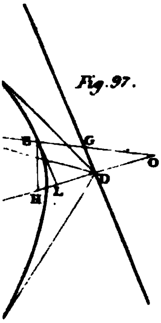

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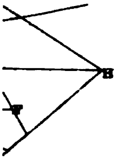

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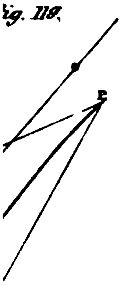

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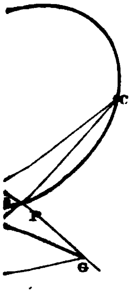

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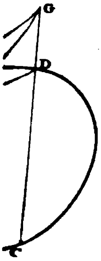

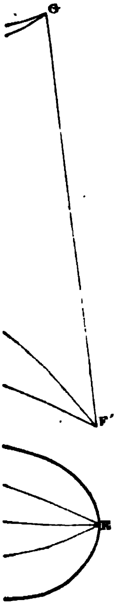

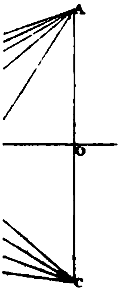
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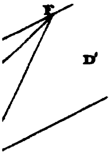
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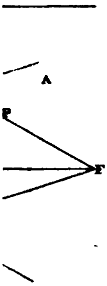
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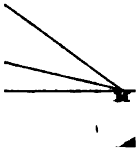
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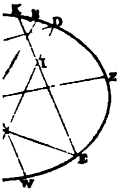

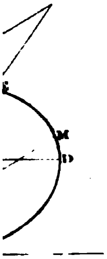

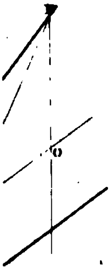

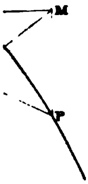

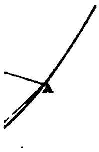

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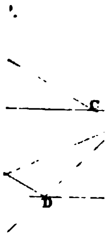

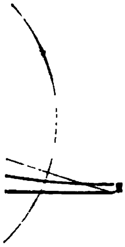

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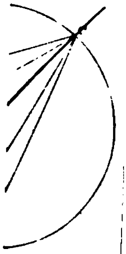

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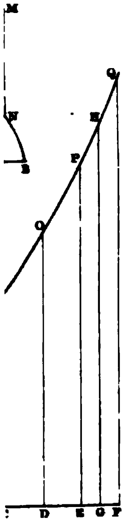

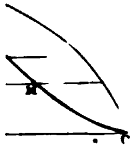

LATE XXII.

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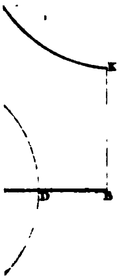

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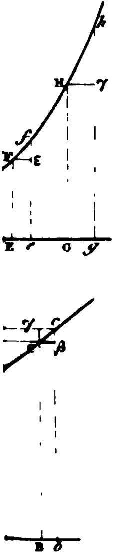

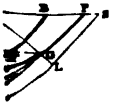

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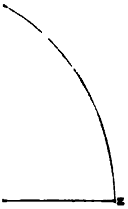

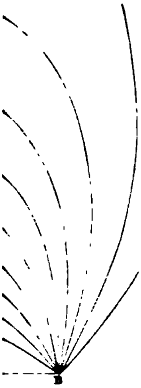

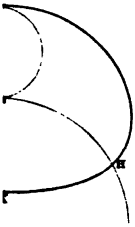

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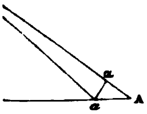

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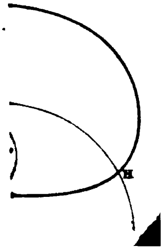

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Fig. 217.